On (enriched) $L$-fuzzy Topologies: Decomposition Theorem

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Abstract:
To our knowledge, the existed decomposition theorems for $L$-fuzzy topologies are only available in the case that the lattices to be completely distributive complete lattice. In this paper, considering $L$ to be an arbitrary complete Heyting algebra, a decomposition theorem for (enriched) $L$-fuzzy topologies is presented. Said precisely, it is proved that an (enriched) $L$-fuzzy topology can be represented as a family of (stratified) $L$-topologies with some left-continuous condition.

Key words: Enriched $L$-fuzzy topology, stratified $L$-topology, decomposition theorem

1. Introduction

The notion of fuzzy topologies was first initiated by Chang [2]. He defined a fuzzy topology on a set $X$ as a crisp subset of $I$-power set $I^X$. Later, Goguen [4] generalized this notion from $I$ to arbitrary complete lattice $L$. Thus this kind of fuzzy topology is often called Chang-Goguen $L$-topology, or $L$-topology for short. In a completely different direction, H"ohle [5] defined a fuzzy topology on a set $X$ as a fuzzy subset of the power set $2^X$. Ying [11] studied H"ohle’s fuzzy topology from a logical point of view and called it fuzzifying topology Kubiak [7] and Šostak [9] extended H"ohle’s definition and defined a fuzzy topology on a set $X$ as a fuzzy subset of the $L$-power set $L^X$. This kind of fuzzy topology is usually called $L$-fuzzy topology. The decomposition theorems for $L$-fuzzy topologies are important content in the theory of fuzzy topologies. When $L$ to be a completely distributive complete lattice, we obtain the satisfying results. That is, an $L$-fuzzy topology can decompose a family of $L$-topologies with some additional condition [3,12,13]. But, to our regret, these decompose theorems can not be generalized to the more general lattice contexts. In this paper, we
shall present a decomposition theorem for $L$-fuzzy topology in the case of that $L$ to be complete Heyting algebra. Said precisely, we shall prove that an (enriched) $L$-fuzzy topology can decompose a family of (stratified) $L$-topologies with some left-continuous condition. The enriched $L$-fuzzy topology and stratified $L$-topology means they contain all the constant-valued fuzzy sets.

Let $L$ to be a complete lattice and $a, b \in L$. Then we say “$a$ is wedge below $b$” (resp., “$a$ is way below $b$”), in symbol $a \ll b$ (resp., $a \ll b$) if for each (directed) subset $B \subseteq L$, $b \leq \bigvee B$, implies $a \leq d$ for some $d \in B$. For a completely distributive complete lattice $L$, it possesses the following properties:

1. $a \ll b$ implies that there exists $c \in L$ such that $a \ll c \ll b$; $a \ll b$ implies that there exists $c \in L$, such that $a \ll c \ll b$.
2. $a = \bigvee \{b : b \ll a\} = \bigvee \{b : b \ll a\}$.

Let $\delta$ be a nonempty subset of $L^X$, then it is said to be an $L$-topology on $X$ if $\delta$ satisfies the following conditions:

(T1) $0, 1 \in \delta$.

(T2) $A, B \in \delta \Rightarrow (A \land B) \in \delta$.

(T3) $\forall t \in T, A_i \in \delta \Rightarrow \bigvee_{i \in T} A_i \in \delta$.

The pair $(X, \delta)$ is called an $L$-topological space. Then $\delta$ is said to be stratified if for any $a \in L$, we have $a \in \delta$.

Let $(X, \delta)$ and $(Y, \varepsilon)$ be $L$-topological spaces. The mapping $f : X \to Y$ is said to be a continuous mapping from $(X, \delta)$ to $(Y, \varepsilon)$ if $f_{\varepsilon}^{-}(\lambda) = (\lambda \circ f) \in \delta$ for any $\lambda \in \varepsilon$.

We use the symbol $L$-Top to denote the category composed by $L$-topological space and continuous mapping.

Let $\tau : L^X \to L$ be a fuzzy subset of $L^X$. Then we call $\tau$ an $L$-fuzzy topology on $L^X$ if $\tau$ satisfies the following conditions:

(FT1) $\tau(0_X) = \tau(1_X) = 1$.

(FT2) $\tau(\lambda \land \mu) \geq \tau(\lambda) \land \tau(\mu)$.

(FT3) $\tau(\bigvee_{i \in T} \lambda_i) \geq \bigvee_{i \in T} \tau(\lambda_i)$.

The pair $(X, \tau)$ is called an $L$-fuzzy topological space. Then $\tau$ is said to be enriched if it further satisfies: (FT4) $\forall a \in L$, $\tau(a) = 1$.

Let $(X, \delta)$ and $(Y, \varepsilon)$ be $L$-fuzzy topological spaces. The mapping $f : X \to Y$ is said to be a continuous mapping from $(X, \delta)$ to $(Y, \varepsilon)$ if $\sigma(\lambda) \leq \tau(f_{\varepsilon}^{-}(\lambda))$ for any $\lambda \in L^Y$. Let $L$-FTop denote category whose object are $L$-fuzzy topological spaces and morphisms are continuous mapping.

2. The main conclusions

Let \( LT(X) \) (resp., \( LFT(X) \)) denotes all \( L \)-topologies (resp., \( L \)-fuzzy topologies) on a nonempty set \( X \). Then \( LT(X) \) (resp., \( LFT(X) \)) constitute a complete lattice under the inclusion order (i.e., point-by-point order).

**Definition 1.1.** Let \( X \) be a nonempty set. Then the mapping \( \Gamma : L \rightarrow LT(X) \) is called a layer \( L \)-fuzzy topology if it satisfies the following conditions:

1. \( a \leq b \Rightarrow \Gamma(b) \subseteq \Gamma(a) \),
2. \( \Gamma(0) \) is discrete \( L \)-topology.

The pair \((X, \Gamma)\) is called a layer \( L \)-fuzzy topological space. Further, if each \( \Gamma(a) \) is stratified \( L \)-topology then \( \Gamma \) is called stratified layer \( L \)-fuzzy topology.

A layer (stratified) \( L \)-fuzzy topological space \((X, \Gamma)\) is said to be left continuous if it additionally satisfies the following condition:

3. \( \lambda \in \Gamma(a) \) iff there exists \( B \subset L \) such that \( \lambda \in \Gamma(b) \) and \( \bigvee B \geq a \) for any \( b \in B \).

Let \( CLT(X) \) denote all the layer \( L \)-fuzzy topology on \( X \).

Let \( \left\{ \Gamma(a) \right\} \subseteq CLT(X) \), \( \Xi \in CLT(Y) \). A mapping \( f : X \rightarrow Y \) is said to be a continuous mapping between layer \( L \)-fuzzy topologies \((X, \Gamma)\) and \((Y, \Xi)\) if \( f \) is continuous between \( L \)-topological spaces \((X, \Gamma(a))\) and \((Y, \Xi(a))\) for any \( a \in L \). Denote \( L\text{-CTop} \) as category constituting by all the above objects and continuous mapping.

**Proposition 1.1.** Let \((X, \tau)\) be an \( L \)-fuzzy topological space. Then for any \( a \in L \), the set \( \tau_a = \left\{ \lambda \in L^X : \tau(\lambda) \geq a \right\} \) is an \( L \)-topology on \( X \). In addition, for the mapping \( \Gamma^\tau : L \rightarrow LT(X) \) defined by \( \forall a \in L \, , \, \Gamma^\tau(a) = \tau_a = \left\{ \lambda \in L^X : \tau(\lambda) \geq a \right\} \), we have \( \Gamma^\tau \in CLT(X) \).

**Proof.** It is sufficient to check that \( \Gamma^\tau \) satisfies the conditions (1)-(3) in Definition 1.1.

1. Let \( a, b \in L \, , \, a \leq b \). By the definition of \( \Gamma^\tau \), we have \( \Gamma^\tau(a) = \tau_a \, , \, \Gamma^\tau(b) = \tau_b \).

Then it follows that \( \tau_b \subseteq \tau_a \) and so, \( \Gamma^\tau(b) \subseteq \Gamma^\tau(a) \).

2. It is easily seen that \( \Gamma^\tau(0) = \tau_0 = \left\{ \lambda \in L^X : \tau(\lambda) \geq 0 \right\} = L^X \), i.e. \( \Gamma^\tau(0) \) is the discrete \( L \)-topology on \( X \).

3. Necessity. Note that if \( \lambda \in \Gamma^\tau(a) \), then \( \lambda \in \tau_a \), i.e. \( \tau(\lambda) \geq a \). So, the left-continuous condition holds by taking \( B = \{a\} \).

Sufficiency: Suppose \( B \subset L \) satisfies the condition (3) of Definition 1.1. For all \( b \in L \, , \, \lambda \in \Gamma^\tau(b) \), we have \( \tau(\lambda) \geq b \), then \( \tau(\lambda) \geq \bigvee \{b \in B \} \geq a \). It follows that \( \lambda \in \Gamma^\tau(a) \) by the definition of \( \Gamma^\tau \).

A combination of the above, we have \( \Gamma^\tau \in CLT(X) \).
Proposition 1.2. If the mapping $f : (X, \tau) \to (Y, \sigma)$ is continuous in $L$-$\text{FTop}$, then $f : (X, \Gamma^{r}) \to (Y, \Gamma^\sigma)$ is continuous in $L$-$\text{CTop}$.

Proof. Note that we need to prove the following inclusion $\forall a \in L, \lambda \in \Gamma^\sigma(a) \Rightarrow \lambda \circ f \in \Gamma^r(a)$. Actually, for each $\lambda \in \Gamma^\sigma(a)$, by the continuity of the mapping $f : (X, \tau) \to (Y, \sigma)$, we obtain that $\tau(\lambda \circ f) \geq \sigma(\lambda) \geq a$, i.e., $\lambda \circ f \in \tau_a$. Then it follows that $\lambda \circ f \in \Gamma^r(a)$ by $\Gamma^r(a) = \tau_a$.

Proposition 1.3. Let $\Gamma \in \text{CLT}(X)$. Then the mapping $r^\Gamma : L^X \to L$ defined by

$$\forall \lambda \in L^X, \quad r^\Gamma(\lambda) = \lor \{a \in L : \lambda \in \Gamma(a)\},$$

is an $L$-$\text{fuzzy topology}$ on $X$.

Proof. (FT1) For any $a \in L$, then $0_X, 1_X \in \Gamma(a)$, thus $\tau^\Gamma(1_X) = \tau^\Gamma(0_X) = 1$.

(FT2) Let $\lambda, \mu \in L^X$. Because $a \leq b \Rightarrow \Gamma(b) \subseteq \Gamma(a)$, then $\tau^\Gamma(\lambda) \land \tau^\Gamma(\mu) = (\lor \{a : \lambda \in \Gamma(a)\}) \land (\lor \{b : \mu \in \Gamma(b)\}) = \lor \{a \land b : \lambda \in \Gamma(a) \land \mu \in \Gamma(b)\} \leq \lor \{a \land b : \lambda \in \Gamma(a) \land \mu \in \Gamma(b)\} = \lor \{c : \lambda \land \mu \in \Gamma(c)\} = r^\Gamma(\lambda \land \mu)$, where, the second inequality holds because $\Gamma(a \land b)$ is an $L$-$\text{topology}$, thus it is closed w.r.t. finite intersections.

(FT3) For any $\{\lambda_t : t \in T\} \subseteq L^X$, let $a = \land_{t \in T} \tau^\Gamma(\lambda_t) = \land_{t \in T} (\lor \{a_t : \lambda_t \in \Gamma(a_t)\})$. Then for any $t \in T$, we get $\lor \{a_t : \lambda_t \in \Gamma(a_t)\} \geq a$, and thus $\lambda_t \in \Gamma(a)$ by Definition 1.1 (3). Furthermore, because $\Gamma(a)$ is an $L$-$\text{topology}$, thus $\lor_{t \in T} \lambda_t \in \Gamma(a)$. By the definition of $\tau^\Gamma$, we obtain $\tau^\Gamma\left(\lor_{t \in T} \lambda_t\right) \geq a = \land_{t \in T} \tau(\lambda_t)$.

Proposition 1.4. If the mapping $f : (X, \Gamma) \to (Y, \Xi)$ is continuous in $L$-$\text{CTop}$, then $f : (X, \tau^\Gamma) \to (Y, \tau^\Xi)$ is continuous in $L$-$\text{FTop}$.

Proof. Suppose $\tau^\Xi(\lambda) = b$, for any $\lambda \in L^r$, then $\lambda \in \Xi(b)$. For $f$ is continuous, so $\lambda \circ f \in \Gamma(b)$, then $\tau^\Gamma(\lambda \circ f) \geq b = \tau^\Xi(\lambda)$, i.e. $f : (X, \tau^\Gamma) \to (Y, \tau^\Xi)$ is continuous.

Proposition 1.5 (Decomposition theorem). Let $\Gamma \in \text{CLT}(X), \tau \in \text{LFT}(X)$, then

(1) $\tau^{\Gamma'} = \tau$,

(2) $\tau^{\Gamma'} = \Gamma$, i.e. there exists a one-to-one correspondence between CLT$(X)$ and LFT$(X)$. 
Proof. (1) For all \( \lambda \in L^X \), we have \( \tau^\Gamma (\lambda) = \{ a \in L : \lambda \in \Gamma^c (a) \} = \{ a \in L : \tau (\lambda) \geq a \} = \tau (\lambda) \)

(2) Note that we only need to check \( \Gamma^c (a) = \Gamma (a) \) for any \( a \in L \). For one thing

\[
\Gamma^c (a) = (\tau^\Gamma)_a = \{ \lambda \in L : \tau^\Gamma (\lambda) \geq a \} = \{ \lambda \in L : \lambda \in \Gamma(a) \} = \Gamma (a) .
\]

For another, for any \( \lambda \in (\tau^\Gamma)_a \), one has \( a \leq \tau^\Gamma (\lambda) = \bigvee \{ b \in L : \lambda \in \Gamma(b) \} \). According to Definition 1.1

(3), we have \( \lambda \in \Gamma (a) \). A combination of the above we have \( \Gamma^c (a) = \Gamma (a) \).

Corollary 1.1. Categories \( L \)-CTop isomorphic to \( L \)-FTop.

\[
\Phi ( (X, \Gamma)) = (X, \tau^\Gamma) : \Phi^{-1} ((X, \tau)) = (X, \Gamma^c) .
\]

Remark. Some decomposition theorems have been presented in [3,12,13] in the case of \( L \) being completely distributive lattice. It is easily seen that these theorems rely on the properties (1) and (2) posed by completely distributive lattice. Thus it seems that those decomposition theorems cannot be extended to the more general lattice contexts. In addition, we can obtain a similar decomposition theorem for the enriched \( L \)-fuzzy topologies.

References

On Pairwise Intuitionistic Fuzzy Resolvable (Irresolvable) Spaces

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Abstract:
In this paper, we introduce and study the concepts of intuitionistic fuzzy resolvability, intuitionistic fuzzy irresolvability and intuitionistic fuzzy open hereditarily irresolvability in intuitionistic fuzzy bitopological spaces.

Key words and phrases:
Intuitionistic fuzzy bitopology, pairwise intuitionistic fuzzy resolvable spaces.

1. Introduction

After the introduction of fuzzy sets by Zadeh [5], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [1] is one among them. Using the notion of intuitionistic fuzzy sets, Coker [3] introduced the notion of intuitionistic fuzzy topological spaces. In this paper, we introduce and study the concepts of intuitionistic fuzzy resolvability, intuitionistide fuzzy irresolvability and intuitionistic fuzzy open hereditarily irresolvability in intuitionistic fuzzy bitopological sapces.

2. Preliminaries
Definition 2.1 [1]. Let $A$ be a nonempty fixed set. An intuitionistic fuzzy set (IFS, for short) $A$ is an object having the form $A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}$ where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote respectively the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\gamma_A(x)$) of each element $x \in X$ to the set $A$, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

Obviously, every fuzzy set $A$ on a nonempty set $X$ is an IFS having the form $A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}$.

Definition 2.2 [1]. Let $X$ be a nonempty set and let the IFSs $A$ and $B$ in the form $A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}$ and $B = \{ (x, \mu_B(x), \gamma_B(x)) : x \in X \}$. Let $\{ A_j : j \in J \}$ be an arbitrary family of IFSs in $(X, \tau)$.

Then,

1. $A \subseteq B$ if and only if $\forall x \in X \left[ \mu_A(x) \leq \mu_B(x) \text{ and } \gamma_A(x) \geq \gamma_B(x) \right]$;
2. $1 - A = \{ (x, \gamma_A(x), \mu_A(x)) : x \in X \}$;
3. $\bigcap A_j = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}$;
4. $\bigcup A_j = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}$;
5. $1 = \{ (x, 1, 0) : x \in X \}$ and $0 = \{ (x, 0, 1) : x \in X \}$.

Definition 2.3. An intuitionistic fuzzy topology [3] (IFT, for short) on a nonempty set $X$ is a family $\tau$ of IFSs in $X$ satisfying the following axioms:

(i) $0, 1 \in \tau$;
(ii) $A_1 \cap A_2 \in \tau$ for every $A_1, A_2 \in \tau$;
(iii) $\bigcup A_j \in \tau$ for any $\{ A_j : j \in J \} \subseteq \tau$.

In this case, the ordered pair $(X, \tau)$ is called an intuitionistic fuzzy topological space (IFTS, for short) and each IFS in $\tau$ is known as an intuitionistic fuzzy open set (IFOS, for short) in $X$. The complement of an intuitionistic fuzzy open set is called an intuitionistic fuzzy closed set (IFCS, for short). The family of all IFOSs (resp. IFCSs) of $(X, \tau)$ is denoted by $\text{IFO}(X)$ (resp. $\text{IFC}(X)$).

Definition 2.4. A triple $(X, \tau_1, \tau_2)$, where $X$ is a nonempty set, $\tau_1$ and $\tau_2$ are two arbitrary intuitionistic fuzzy topologies on $X$ is called the intuitionistic fuzzy bitopological space (for short, IFBTS).

Definition 2.5 [3]. Let $(X, \tau)$ be an IFTS and $A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}$ be an IFS in $X$. Then the intuitionistic fuzzy interior and the intuitionistic fuzzy closure
of $A$ is defined by $\text{Int}_r(A) = \bigcup \{ G \setminus G \text{ is an IFOS in } X \text{ and } G \subseteq A \}$ and $\text{Cl}_r(A) = \bigcap \{ G \setminus G \text{ is an IFCS in } X \text{ and } G \supseteq A \}$.

**Remark 2.6.** For any IFS $A$ in $(X, \tau)$, we have, $\text{Cl}_r(1-A) = 1 - \text{Int}_r(A)$, $\text{Int}_r(1-A) = 1 - \text{Cl}_r(A)$.

3. **Pairwise intuitionistic fuzzy resolvable and intuitionistic fuzzy irresolvable spaces**

**Definition 3.1.** An intuitionistic fuzzy bitopological space $(X, \tau_1, \tau_2)$ is called a pairwise intuitionistic fuzzy resolvable space if there exists a $\tau_1$-intuitionistic fuzzy dense set $\lambda$ such that $1 - \lambda$ is a $\tau_2$-intuitionistic fuzzy dense set and a $\tau_2$-fuzzy dense set $\mu$ such that $1 - \mu$ is a $\tau_1$-intuitionistic fuzzy dense set. Otherwise $(X, \tau_1, \tau_2)$ is called a pairwise intuitionistic fuzzy irresolvable space.

**Example 3.2.** Let $X = \{a, b\}$ and $A, B$ be intuitionistic fuzzy sets defined by

$$ A = \left\{ x, \left( \begin{array}{c} a \\ 0.3 \\ 0.4 \end{array} \right), \left( \begin{array}{c} a \\ 0.6 \\ 0.5 \end{array} \right) \right\}, \quad B = \left\{ x, \left( \begin{array}{c} a \\ 0.7 \\ 0.8 \end{array} \right), \left( \begin{array}{c} a \\ 0.3 \\ 0.2 \end{array} \right) \right\}. $$

Let $\tau_1 = \{0, A, 1\}$ and $\tau_2 = \{0, B, 1\}$. Then $(X, \tau_1, \tau_2)$ is a pairwise intuitionistic fuzzy resolvable space.

**Example 3.3.** Let $X = \{a, b\}$ and $A, B$ be intuitionistic fuzzy sets defined by

$$ A = \left\{ x, \left( \begin{array}{c} a \\ 0.6 \\ 0.5 \end{array} \right), \left( \begin{array}{c} a \\ 0.4 \\ 0.5 \end{array} \right) \right\}, \quad B = \left\{ x, \left( \begin{array}{c} a \\ 0.3 \\ 0.2 \end{array} \right), \left( \begin{array}{c} a \\ 0.5 \\ 0.5 \end{array} \right) \right\}. $$

Let $\tau_1 = \{0, A, 1\}$ and $\tau_2 = \{0, B, 1\}$. Then $(X, \tau_1, \tau_2)$ is a pairwise intuitionistic fuzzy irresolvable space.

**Proposition 3.4.** An intuitionistic fuzzy bitopological space $(X, \tau_1, \tau_2)$ is a pairwise intuitionistic fuzzy irresolvable space.

**Proposition 3.4.** An intuitionistic fuzzy bitopological space $(X, \tau_1, \tau_2)$ is a pairwise intuitionistic fuzzy resolvable space if and only if $(X, \tau_1, \tau_2)$ has a $\tau_1$-intuitionistic fuzzy dense set $\lambda_1$ and $\tau_2$-intuitionistic fuzzy dense set $\lambda_2$ such that $\lambda_1 \leq 1 - \lambda_2$.

**Proof.** Let $(X, \tau_1, \tau_2)$ be a pairwise intuitionistic fuzzy resolvable space. Suppose that for all $\tau_1$-intuitionistic fuzzy dense sets, $\lambda$, and $\tau_2$-intuitionistic fuzzy dense sets $\lambda_1$, $\lambda \leq 1 - \lambda_1$. That is $\lambda_1 > 1 - \lambda$. Then $\text{Cl}_{\tau_1}(\lambda) > \text{Cl}_{\tau_1}(1 - \lambda)$. Now $\lambda_1$ is a $\tau_1$-intui-
An intuitionistic fuzzy bitopological space \((X, \tau_1, \tau_2)\) is pairwise intuitionistic fuzzy irresolvable if and only if there exists a \(\tau_1\)-intuitionistic fuzzy dense set and a \(\tau_2\)-intuitionistic fuzzy dense set such that \(\lambda_1 \leq 1 - \lambda_2\). Conversely, suppose that \((X, \tau_1, \tau_2)\) is a \(\tau_1\)-intuitionistic fuzzy irresolvable space. Suppose that \((X, \tau_1, \tau_2)\) is a pairwise intuitionistic fuzzy irresolvable space. Then, for all \(\tau_i\)-intuitionistic fuzzy dense sets \(\lambda_i\) and \(\lambda_2\), we have \(\text{Cl}_{\tau_i}(1 - \lambda_i) \neq 1\). That is there exists a \(\tau_2\)-intuitionistic fuzzy closed set \(\mu\) in \((X, \tau_1, \tau_2)\) such that \((1 - \lambda_1) < \mu < 1\). Then \((1 - \mu) < \lambda_1 < 1\) and \((1 - \mu) < \lambda_2 \leq 1 - \lambda_2\) implies that \((1 - \mu) < 1 - \lambda_2 \Rightarrow \lambda_1 < \mu < 1\), which is a contradiction to \(\lambda_1\) being a \(\tau_2\)-intuitionistic fuzzy dense set in \((X, \tau_1, \tau_2)\).

Hence \((X, \tau_1, \tau_2)\) is a pairwise intuitionistic fuzzy irresolvable space.

**Proposition 3.6.** An intuitionistic fuzzy bitopological space \((X, \tau_1, \tau_2)\) is pairwise intuitionistic fuzzy irresolvable if and only if there exists a \(\tau_1\)-intuitionistic fuzzy dense set \(\lambda_1\), \(\text{Int}_{\tau_i}(\lambda_1) \neq 0\) and for a \(\tau_2\)-intuitionistic fuzzy dense set \(\lambda_2\), \(\text{Int}_{\tau_2}(\lambda_2) \neq 0\).

**Proof.** Let \(\lambda_1\) be a \(\tau_1\)-intuitionistic fuzzy dense set and \(\lambda_2\) a \(\tau_2\)-intuitionistic fuzzy dense set in \((X, \tau_1, \tau_2)\). Since \((X, \tau_1, \tau_2)\) is a pairwise intuitionistic fuzzy irresolvable space, \(1 - \lambda_1\) is not a \(\tau_2\)-intuitionistic fuzzy dense set and \(1 - \lambda_2\) is not a \(\tau_1\)-intuitionistic fuzzy dense set in \((X, \tau_1, \tau_2)\). That is, \(\text{Cl}_{\tau_2}(1 - \lambda_1) \neq 1\) and \(\text{Cl}_{\tau_1}(1 - \lambda_2) \neq 1\). Then, we have \(1 - \text{Int}_{\tau_2}(\lambda_1) \neq 1\) and \(1 - \text{Int}_{\tau_1}(\lambda_2) \neq 1\). Hence we have \(\text{Int}_{\tau_1}(\lambda_1) \neq 0\) and \(\text{Int}_{\tau_2}(\lambda_2) \neq 0\). Conversely, let \(\lambda_1\) be a \(\tau_1\)-intuitionistic fuzzy dense set and \(\lambda_2\) a \(\tau_2\)-intuitionistic fuzzy dense set in \((X, \tau_1, \tau_2)\). By hypothesis, \(\text{Int}_{\tau_1}(\lambda_1) \neq 0\) and \(\text{Int}_{\tau_2}(\lambda_2) \neq 0\). Then \(1 - \text{Int}_{\tau_2}(\lambda_1) \neq 1\) and \(1 - \text{Int}_{\tau_1}(\lambda_2) \neq 1\) implies that \(\text{Cl}_{\tau_2}(1 - \lambda_1) \neq 1\) and \(\text{Cl}_{\tau_1}(1 - \lambda_2) \neq 1\). That is, \(\text{Cl}_{\tau_1}(\lambda_1) = 1\) implies that \(\text{Cl}_{\tau_2}(1 - \lambda_1) \neq 1\) and \(\text{Cl}_{\tau_1}(\lambda_2) = 1\) implies that \(\text{Cl}_{\tau_2}(1 - \lambda_2) \neq 1\). Hence \((X, \tau_1, \tau_2)\) is a pairwise intuitionistic fuzzy irresolvable space.
\(\lambda\) and \(\tau_2\)-intuitionistic fuzzy dense set \(\lambda_2\) in \((X,\tau_1,\tau_2)\) such that \(\text{Int}_{\tau_1}(\lambda) = 0\) and \(\text{Int}_{\tau_2}(\lambda_2) = 0\).

**Proof.** The proof follows from Proposition 3.5.

**Definition 3.7.** An intuitionistic fuzzy bitopological space \((X,\tau_1,\tau_2)\) is called a pairwise intuitionistic fuzzy submaximal if each \(\tau_1\)-intuitionistic fuzzy dense set is a \(\tau_2\)-intuitionistic fuzzy open set and each \(\tau_2\)-intuitionistic fuzzy dense set is a \(\tau_1\)-intuitionistic fuzzy open set.

**Example 3.8.** Let \(X = \{a,b\}\) and \(A, B\) be intuitionistic fuzzy sets defined by

\[
A = \left\{ x, \begin{pmatrix} a \cdot 0.3 & b \cdot 0.4 \\ a \cdot 0.3 & b \cdot 0.6 \\ a \cdot 0.7 & b \cdot 0.5 \\ a \cdot 0.8 & b \cdot 0.5 \end{pmatrix} \right\}, \quad B = \left\{ x, \begin{pmatrix} a \cdot 0.3 & b \cdot 0.2 \\ a \cdot 0.7 & b \cdot 0.8 \\ a \cdot 0.3 & b \cdot 0.2 \end{pmatrix} \right\}
\]

\[
C = \left\{ x, \begin{pmatrix} a \cdot 0.4 & b \cdot 0.6 \\ a \cdot 0.4 & b \cdot 0.6 \\ a \cdot 0.1 & b \cdot 0.3 \\ a \cdot 0.3 & b \cdot 0.5 \end{pmatrix} \right\}, \quad D = \left\{ x, \begin{pmatrix} a \cdot 0.4 & b \cdot 0.6 \\ a \cdot 0.4 & b \cdot 0.6 \\ a \cdot 0.1 & b \cdot 0.3 \\ a \cdot 0.3 & b \cdot 0.5 \end{pmatrix} \right\}
\]

Let \(\tau_1 = \{0,A,D,1\}\) and \(\tau_2 = \{0,B,C,1\}\). Then \((X,\tau_1,\tau_2)\) is a pair-wise intuitionistic fuzzy submaximal space.

**Proposition 3.9.** If an intuitionistic fuzzy bitopological space \((X,\tau_1,\tau_2)\) is pairwise intuitionistic fuzzy submaximal, then it is pairwise intuitionistic fuzzy irresolvable.

**Proof.** Let \((X,\tau_1,\tau_2)\) be a pairwise intuitionistic fuzzy submaximal space. Let \(\lambda\) be a \(\tau_1\)-intuitionistic fuzzy dense set in \((X,\tau_1,\tau_2)\) and \(\mu\) be a \(\tau_2\)-intuitionistic fuzzy dense set in \((X,\tau_1,\tau_2)\). Since \((X,\tau_1,\tau_2)\) is pairwise intuitionistic fuzzy submaximal, \(\lambda\) is \(\tau_2\)-intuitionistic fuzzy open and \(\mu\) is \(\tau_1\)-intuitionistic fuzzy open. Then we have \(\text{Int}_{\tau_1}(\lambda) = \lambda \neq 0\) and \(\text{Int}_{\tau_2}(\mu) = \mu \neq 0\). Hence by the Proposition 3.5, \((X,\tau_1,\tau_2)\) is a pairwise intuitionistic fuzzy irresolvable space.

**Definition 3.10.** An intuitionistic fuzzy bitopological space \((X,\tau_1,\tau_2)\) is called a pairwise intuitionistic fuzzy open hereditarily irresolvable if \(\text{Int}_{\tau_1}(\lambda) = \lambda \neq 0\) for any \(\tau_1\)-intuitionistic fuzzy open set \(\lambda\) implies that \(\text{Int}_{\tau_2}(\lambda) \neq 0\) and \(\text{Int}_{\tau_1}(\mu) \neq 0\) for any \(\tau_2\)-intuitionistic fuzzy open set \(\mu\) implies that \(\text{Int}_{\tau_1}(\mu) \neq 0\).

**Proposition 3.11.** If an intuitionistic fuzzy bitopological space \((X,\tau_1,\tau_2)\) is pairwise intuitionistic fuzzy open hereditarily irresolute, then it is pairwise intuitionistic fuzzy irresolvable.

**Proof.** Let \(\lambda\) be a \(\tau_1\)-intuitionistic fuzzy dense set and \(\mu\) be a \(\tau_2\)-intuitionistic fuzzy dense set in \((X,\tau_1,\tau_2)\). Now \(\text{Cl}_{\tau_1}(\lambda) = 1\) implies that \(\text{Int}_{\tau_2}(\text{Cl}_{\tau_1}(\lambda)) = \text{Int}_{\tau_1}(1) = 1 \neq 0\) and
\( \text{Cl}_{\tau} (\mu) = 1 \) implies that \( \text{Int}_{\tau} \text{Cl}_{\tau} (\mu) = \text{Int}_{\tau} (1) = 1 \neq 0 \). Since \((X, \tau_1, \tau_2)\) is pairwise intuitionistic fuzzy open hereditarily irresolvable, \( \text{Int}_{\tau_1} (\lambda) \neq 0 \) and \( \text{Int}_{\tau_1} (\mu) \neq 0 \). Hence for a \( \tau_1 \)-intuitionistic fuzzy dense set \( \lambda \), \( \text{Int}_{\tau_1} (\lambda) \neq 0 \) and for a \( \tau_2 \)-intuitionistic fuzzy dense set \( \mu \), \( \text{Int}_{\tau_1} (\mu) \neq 0 \). Therefore, by Proposition 3.5 \((X, \tau_1, \tau_2)\) is a pairwise intuitionistic fuzzy irresolvable space.

**Remark 3.12.** It is clear that every pairwise intuitionistic fuzzy open hereditarily irresolvable space is pairwise intuitionistic fuzzy irresolvable. However, the converse need not be true.

**Example 3.13.** Let \( X = \{a, b\} \) and \( A, B \) be intuitionistic fuzzy sets defined by
\[
A = \left\{ x : \left( \frac{a}{0.6}, \frac{b}{0.5} \right), \left( \frac{a}{0.4}, \frac{b}{0.5} \right) \right\}, \quad B = \left\{ x : \left( \frac{a}{0.3}, \frac{b}{0.2} \right), \left( \frac{a}{0.6}, \frac{b}{0.5} \right) \right\}
\]
Let \( \tau_1 = \{0, A, 1\} \) and \( \tau_2 = \{0, B, 1\} \). Then the intuitionistic fuzzy bitopological space \((X, \tau_1, \tau_2)\) is pairwise intuitionistic fuzzy irresolvable but not pairwise intuitionistic fuzzy open hereditarily irresolvable.

**Proposition 3.14.** If an intuitionistic fuzzy bitopological space \((X, \tau_1, \tau_2)\) is pairwise intuitionistic fuzzy open hereditarily irresolvable and if \( \text{Int}_{\tau_1} (\lambda) = 0 \) for a \( \tau_1 \)-intuitionistic fuzzy open set \( \lambda \) and \( \text{Int}_{\tau_1} (\mu) = 0 \) for a \( \tau_2 \)-intuitionistic fuzzy open set \( \mu \), then \( \text{Int}_{\tau_1} (\text{Cl}_{\tau_1} (\lambda)) = 0 \) and \( \text{Int}_{\tau_1} (\text{Cl}_{\tau_1} (\mu)) = 0 \).

**Proof.** Let \( \lambda \neq 0 \) be a \( \tau_1 \)-intuitionistic fuzzy open set in \((X, \tau_1, \tau_2)\) such that \( \text{Int}_{\tau_1} (\lambda) = 0 \) and \( \mu \neq 0 \) be a \( \tau_2 \)-intuitionistic fuzzy open set in \((X, \tau_1, \tau_2)\) such that \( \text{Int}_{\tau_1} (\mu) = 0 \). We claim that \( \text{Int}_{\tau_1} (\text{Cl}_{\tau_1} (\lambda)) = 0 \) and \( \text{Int}_{\tau_1} (\text{Cl}_{\tau_1} (\mu)) = 0 \). Suppose that \( \text{Int}_{\tau_1} (\text{Cl}_{\tau_1} (\lambda)) \neq 0 \) and \( \text{Int}_{\tau_1} (\text{Cl}_{\tau_1} (\mu)) \neq 0 \). Since \((X, \tau_1, \tau_2)\) is pairwise intuitionistic fuzzy open hereditarily irresolvable, we have \( \text{Int}_{\tau_1} (\text{Cl}_{\tau_1} (\lambda)) \neq 0 \) for any \( \tau_1 \)-intuitionistic fuzzy open set \( \lambda \) implies that \( \text{Int}_{\tau_1} (\lambda) \neq 0 \) and \( \text{Int}_{\tau_1} (\text{Cl}_{\tau_1} (\mu)) \neq 0 \) for any \( \tau_2 \)-intuitionistic fuzzy open set \( \mu \) implies that \( \text{Int}_{\tau_1} (\mu) \neq 0 \), which is a contradiction to our hypothesis. Hence we must have \( \text{Int}_{\tau_1} (\text{Cl}_{\tau_1} (\lambda)) = 0 \) and \( \text{Int}_{\tau_1} (\text{Cl}_{\tau_1} (\mu)) = 0 \).

**Definition 3.15.** An intuitionistic fuzzy bitopological space \((X, \tau_1, \tau_2)\) is called a pairwise intuitionistic fuzzy hyperconnected if for every nonzero \( \tau_1 \)-intuitionistic fuzzy open set \( \lambda \) and a \( \tau_2 \)-intuitionistic fuzzy open set \( \mu \) such that \( \lambda + \mu > 1 \).
Proposition 3.16. An intuitionistic fuzzy bitopological space \((X, \tau_1, \tau_2)\) is pairwise intuitionistic fuzzy hyperconnected if and only if for any nonzero intuitionistic fuzzy set \(\lambda\) in \((X, \tau_1, \tau_2)\) either \(\text{Cl}_{\tau_1}(\lambda) = 1\) or \(\text{Cl}_{\tau_2}(1-\lambda) = 1\).

Proof. Let \((X, \tau_1, \tau_2)\) be a pairwise intuitionistic fuzzy hyperconnected space. Then for every \(\tau_1\)-intuitionistic fuzzy open set \(\lambda(\neq 0)\) and a \(\tau_2\)-intuitionistic fuzzy open set \(\mu(\neq 0)\) such that \(\lambda + \mu > 1\). Suppose that for an intuitionistic fuzzy set \(\gamma\), \(\text{Cl}_{\tau_1}(\gamma) \neq 1\) and \(\text{Cl}_{\tau_2}(1-\gamma) \neq 1\). Now \(\text{Cl}_{\tau_1}(\gamma) \neq 1\) implies that \(1-\text{Cl}_{\tau_1}(\gamma) \neq 0\). Then \(\text{Int}_{\tau_1}(1-\gamma) \neq 0\). Hence there exists a \(\tau_1\)-intuitionistic fuzzy open set \(\delta\) in \((X, \tau_1, \tau_2)\) such that \(\delta \leq 1-\gamma\). Also \(\text{Cl}_{\tau_1}(1-\gamma) \neq 1\) implies that there exists a \(\tau_2\)-intuitionistic fuzzy closed set \(\eta\) in \((X, \tau_1, \tau_2)\) such that \(1-\gamma \leq \eta < 1\). Hence \(\delta \leq 1-\gamma \leq \eta\) implies that \(\delta \leq \eta\). That is \(\delta \leq 1-(1-\eta)\). Then \(\delta+(1-\eta) \leq 1\). That is, for a nonzero \(\tau_1\)-intuitionistic fuzzy open set \(\delta\) and a \(\tau_2\)-intuitionistic fuzzy open set \(1-\eta\) in \((X, \tau_1, \tau_2)\) we have \(\delta+(1-\eta) \leq 1\), which is a contradiction to our hypothesis. Hence we have either \(\text{Cl}_{\tau_1}(\gamma) = 1\) or \(\text{Cl}_{\tau_1}(1-\gamma) = 1\) for a nonzero intuitionistic fuzzy set \(\gamma\) in \((X, \tau_1, \tau_2)\). Conversely, let \(\text{Cl}_{\tau_1}(\gamma) = 1\) or \(\text{Cl}_{\tau_1}(1-\gamma) = 1\) for any nonzero intuitionistic fuzzy set \(\gamma\) in \((X, \tau_1, \tau_2)\). Suppose that for every \(\tau_1\)-intuitionistic fuzzy open set \(\lambda(\neq 0)\) and a \(\tau_2\)-intuitionistic fuzzy open set \(\mu(\neq 0)\) such that \(\lambda + \mu > 1\). Then \(\lambda + \mu \leq 1\) implies that \(\text{Cl}_{\tau_1}(\lambda) \leq \text{Cl}_{\tau_1}(1-\mu) = 1-\mu\) (since \(\mu\) is a \(\tau_2\)-intuitionistic fuzzy open set \(\Rightarrow 1-\mu\) is \(\tau_2\)-intuitionistic fuzzy closed set in \((X, \tau_1, \tau_2)\)). Hence \(\text{Cl}_{\tau_1}(\lambda) \leq 1-\mu < 1\) (since \(\mu \neq 0\)). That is, \(\text{Cl}_{\tau_1}(\lambda) \neq 1\). Now \(\text{Cl}_{\tau_1}(1-\lambda) = 1-\text{Int}_{\tau_1}(\lambda) = 1-\lambda \neq 1\) (since \(\lambda \neq 0\)). That is, \(\text{Cl}_{\tau_1}(1-\lambda) \neq 1\). Therefore for the nonzero intuitionistic fuzzy set \(1-\lambda\), \(\text{Cl}_{\tau_1}(1-\lambda) \neq 1\) and \(\text{Cl}_{\tau_1}(1-[1-\lambda]) \neq 1\), which is a contradiction to our hypothesis. Hence for every \(\tau_1\)-intuitionistic fuzzy open set \(\lambda(\neq 0)\) and a \(\tau_2\)-intuitionistic fuzzy open set \(\mu(\neq 0)\) such that \(\lambda + \mu > 1\) which implies that \((X, \tau_1, \tau_2)\) is a pairwise intuitionistic fuzzy hyperconnected space.

Proposition 3.17. If an intuitionistic fuzzy bitopological space \((X, \tau_1, \tau_2)\) is pairwise intuitionistic fuzzy hyperconnected, then it is a pairwise intuitionistic fuzzy open hereditarily irresolvable space.

Proof. Let \(\lambda\) be a \(\tau_1\)-intuitionistic fuzzy open set such that \(\text{Int}_{\tau_1}(\text{Cl}_{\tau_1}(\lambda)) \neq 0\) and \(\mu\) a \(\tau_2\)-intuitionistic fuzzy open set such that \(\text{Int}_{\tau_2}(\text{Cl}_{\tau_2}(\mu)) \neq 0\) in a pairwise intuitionistic fuzzy hyperconnected space \((X, \tau_1, \tau_2)\). Then for the nonzero \(\tau_1\)-intuitionistic fuzzy open set \(\lambda\) and for the \(\tau_2\)-intuitionistic fuzzy open set \(\mu\) we have \(\lambda + \mu > 1\). Now \(\lambda + 1-\mu \Rightarrow \text{Int}_{\tau_1}(\lambda) > \text{Int}_{\tau_1}(1-\mu)\). Then \(\text{Int}_{\tau_1}(\lambda) > 1-\text{Cl}_{\tau_1}(\mu) \geq 0\). That is, \(\text{Int}_{\tau_1}(\lambda) \neq 0\). Also,
\[ \lambda + \mu > 1 - \lambda . \] Then \( \text{Int}_{\tau_1}(\mu) > \text{Int}_{\tau_1}(1 - \lambda) = 1 - \text{Cl}_{\tau_1}(\lambda) \geq 0 . \) That is, \( \text{Int}_{\tau_1}(\mu) \neq 0 . \) Therefore \( \text{Int}_{\tau_1}\left(\text{Cl}_{\tau_1}(\lambda)\right) \neq 0 \) for any \( \tau_1 \)-intuitionistic fuzzy open set \( \lambda \) implies that \( \text{Int}_{\tau_1}(\lambda) \neq 0 \) and \( \text{Int}_{\tau_1}\left(\text{Cl}_{\tau_1}(\mu)\right) \neq 0 \) for any \( \tau_2 \)-intuitionistic fuzzy open set \( \mu \) implies that \( \text{Int}_{\tau_2}(\mu) \neq 0 . \) Hence \( (X, \tau_1, \tau_2) \) is a pairwise intuitionistic fuzzy open hereditarily irresolvable space.

**Proposition 3.18.** If an intuitionistic fuzzy bitopological space \( (X, \tau_1, \tau_2) \) is a pairwise intuitionistic fuzzy hyperconnected and pairwise intuitionistic fuzzy irresolvable space, then \( \text{Cl}_{\tau_1}(\lambda) = 1 \) implies that \( \text{Cl}_{\tau_2}(1 - \lambda) \neq 1 \) for any intuitionistic fuzzy set \( \lambda \) in \( (X, \tau_1, \tau_2) \).

**Proof.** Since \( (X, \tau_1, \tau_2) \) is a pairwise intuitionistic fuzzy hyperconnected space, by Proposition 3.16, either \( \text{Cl}_{\tau_1}(\lambda) = 1 \) or \( \text{Cl}_{\tau_1}(1 - \lambda) = 1 \) for any non zero intuitionistic fuzzy set \( \lambda \) in \( (X, \tau_1, \tau_2) \). Suppose \( \text{Cl}_{\tau_1}(\lambda) = 1 \). Now \( (X, \tau_1, \tau_2) \) is a pairwise intuitionistic fuzzy irresolvable space implies that \( \text{Int}_{\tau_1}(\lambda) \neq 0 \). Then \( 1 - \text{Int}_{\tau_1}(\lambda) \neq 1 \). Hence \( \text{Cl}_{\tau_1}(1 - \lambda) \neq 1 \).

**Proposition 3.19.** If an intuitionistic fuzzy bitopological space \( (X, \tau_1, \tau_2) \) is a pairwise intuitionistic fuzzy hyperconnected and pairwise intuitionistic fuzzy irresolvable space, then \( (X, \tau_1, \tau_2) \) is a pairwise intuitionistic fuzzy open hereditarily irresolvable space.

**Proof.** Let \( \lambda \) be a \( \tau_1 \)-intuitionistic fuzzy open set and \( \mu \) be a \( \tau_2 \)-intuitionistic fuzzy open set in \( (X, \tau_1, \tau_2) \). Since \( (X, \tau_1, \tau_2) \) is pairwise intuitionistic fuzzy hyperconnected, by Proposition 3.16, either \( \text{Cl}_{\tau_1}(\lambda) = 1 \) or \( \text{Cl}_{\tau_1}(1 - \lambda) = 1 \). Suppose \( \text{Cl}_{\tau_1}(\lambda) = 1 \). Then by Proposition 3.18, \( \text{Cl}_{\tau_2}(1 - \lambda) \neq 1 \). Now \( \text{Cl}_{\tau_2}(\lambda) = 1 \Rightarrow \text{Int}_{\tau_2}\left(\text{Cl}_{\tau_2}(\lambda)\right) = \text{Int}_{\tau_2}(1) \neq 0 \) and \( (X, \tau_1, \tau_2) \) is a pairwise intuitionistic fuzzy irresolvable space implies that \( \text{Int}_{\tau_2}(\lambda) \neq 0 \). Again since \( (X, \tau_1, \tau_2) \) is intuitionistic fuzzy hyper-connected by Proposition 3.16, either \( \text{Cl}_{\tau_1}(\mu) = 1 \) or \( \text{Cl}_{\tau_1}(1 - \mu) = 1 \) for the \( \tau_2 \)-intuitionistic fuzzy open set \( \mu \). Suppose \( \text{Cl}_{\tau_1}(\mu) = 1 \). Then, by Proposition 3.18 we have \( \text{Cl}_{\tau_1}(1 - \mu) \neq 1 \). Now \( \text{Cl}_{\tau_1}(\mu) = 1 \) implies that \( \text{Int}_{\tau_1}\left(\text{Cl}_{\tau_1}(\mu)\right) = \text{Int}_{\tau_1}(1) \neq 0 \) and \( (X, \tau_1, \tau_2) \) is a pairwise intuitionistic fuzzy irresolvable space implies that \( \text{Int}_{\tau_2}(\mu) \neq 0 \). Hence \( \text{Int}_{\tau_2}\left(\text{Cl}_{\tau_2}(\lambda)\right) = \text{Int}_{\tau_2}(1) \neq 0 \) for any \( \tau_1 \)-intuitionistic fuzzy open set \( \lambda \) implies that \( \text{Int}_{\tau_1}(\lambda) \neq 0 \) and \( \text{Int}_{\tau_2}\left(\text{Cl}_{\tau_2}(\mu)\right) \neq 0 \) for any \( \tau_2 \)-intuitionistic fuzzy open set \( \mu \) implies that \( \text{Int}_{\tau_2}(\mu) \neq 0 \). Therefore, \( (X, \tau_1, \tau_2) \) is a pairwise intuitionistic fuzzy open hereditarily irresolvable space.
Proposition 3.20. For any intuitionistic fuzzy bitopological space $(X, \tau_1, \tau_2)$ the following are equivalent:

(1) $(X, \tau_1, \tau_2)$ is a pairwise intuitionistic fuzzy hyperconnected and pairwise intuitionistic fuzzy irresolvable space.

(2) For every nonzero intuitionistic fuzzy set $\lambda$ in $(X, \tau_1, \tau_2)$ either $\text{Cl}_{\tau_1}(\text{Int}_{\tau_1}(\lambda)) = 1$ or $\text{Cl}_{\tau_2}(\text{Int}_{\tau_2}(1-\lambda)) = 1$.

Proof. (1) $\Rightarrow$ (2) Let $(X, \tau_1, \tau_2)$ be a pairwise intuitionistic fuzzy hyperconnected and pairwise intuitionistic fuzzy irresolvable space. Suppose that $\text{Cl}_{\tau_1}(\text{Int}_{\tau_1}(\lambda)) \neq 1$ and $\text{Cl}_{\tau_2}(\text{Int}_{\tau_2}(1-\lambda)) \neq 1$. Then $1 - \text{Cl}_{\tau_1}(\text{Int}_{\tau_1}(\lambda)) \neq 0$ and $1 - \text{Cl}_{\tau_2}(\text{Int}_{\tau_2}(1-\lambda)) \neq 0$. Hence $\text{Int}_{\tau_1}(\text{Cl}_{\tau_1}(1-\lambda)) \neq 0$ and $\text{Int}_{\tau_2}(\text{Cl}_{\tau_2}(\lambda)) \neq 0$. Since $(X, \tau_1, \tau_2)$ is a pairwise intuitionistic fuzzy open hereditarily irresolvable space, we have $\text{Int}_{\tau_1}(1-\lambda) \neq 0$ and $\text{Int}_{\tau_2}(\lambda) \neq 0$. Then, $1 - \text{Cl}_{\tau_1}(\lambda) \neq 0$ and $1 - \text{Int}_{\tau_1}(\lambda) \neq 0 = 1$. That is, $\text{Cl}_{\tau_1}(\lambda) \neq 1$ and $\text{Cl}_{\tau_2}(1-\lambda) \neq 1$, which is a contradiction to $(X, \tau_1, \tau_2)$ being a pairwise intuitionistic fuzzy hyperconnected space. Hence we have either $\text{Cl}_{\tau_1}(\text{Int}_{\tau_1}(\lambda)) = 1$ or $\text{Cl}_{\tau_2}(\text{Int}_{\tau_2}(1-\lambda)) = 1$.

(2) $\Rightarrow$ (1) Let either $\text{Cl}_{\tau_1}(\text{Int}_{\tau_1}(\lambda)) = 1$ or $\text{Cl}_{\tau_2}(\text{Int}_{\tau_2}(1-\lambda)) = 1$ for every nonzero intuitionistic fuzzy set $\lambda$ in $(X, \tau_1, \tau_2)$. Then $\text{Int}_{\tau_1}(\lambda) \neq 0$ or $\text{Int}_{\tau_2}(1-\lambda) \neq 0$. Therefore for an $\tau_1$-dense intuitionistic fuzzy set $\text{Int}_{\tau_1}(\lambda)$, $\text{Int}_{\tau_1}(\text{Int}_{\tau_1}(\lambda)) = \text{Int}_{\tau_1}(\lambda) \neq 0$ and for a $\tau_2$-dense intuitionistic fuzzy set $\text{Int}_{\tau_2}(1-\lambda)$, $\text{Int}_{\tau_2}(\text{Int}_{\tau_2}(1-\lambda)) = \text{Int}_{\tau_2}(1-\lambda) \neq 0$. Hence by Proposition 3.5, $(X, \tau_1, \tau_2)$ is a pairwise intuitionistic fuzzy irresolvable space. Now $\text{Int}_{\tau_1}(\lambda) \subseteq \lambda$ implies that $\text{Cl}_{\tau_1}\text{Int}_{\tau_1}(\lambda) \subseteq \text{Cl}_{\tau_1}(\lambda)$. Then $1 \leq \text{Cl}_{\tau_1}(\lambda)$. That is, $\text{Cl}_{\tau_1}(\lambda) = 1$. Also $\text{Int}_{\tau_2}(1-\lambda) \subseteq (1-\lambda)$ implies that $\text{Cl}_{\tau_2}\text{Int}_{\tau_2}(1-\lambda) \subseteq \text{Cl}_{\tau_2}(1-\lambda)$. Then $1 \leq \text{Cl}_{\tau_2}(1-\lambda)$. That is, $\text{Cl}_{\tau_2}(1-\lambda) = 1$. Therefore, $\text{Cl}_{\tau_1}(\lambda) = 1$ or $\text{Cl}_{\tau_2}(1-\lambda) = 1$ for any intuitionistic fuzzy set $\lambda$ in $(X, \tau_1, \tau_2)$. Hence by Proposition 3.16, $(X, \tau_1, \tau_2)$ is a pairwise intuitionistic fuzzy hyperconnected space.

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On Generalized Intuitionistic Fuzzy Topology

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Abstract:
The aim of this paper is to present a common approach allowing to obtain families of intuitionistic fuzzy sets in an intuitionistic fuzzy topological space.

Key words and phrases:
Generalized Intuitionistic fuzzy topology, $\gamma$ - intuitionistic fuzzy open set.

1. Introduction

After the introduction of fuzzy sets by Zadeh [7], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [1] is one among them. Using the notion of intuitionistic fuzzy sets, Coker [3] introduced the notion of intuitionistic fuzzy topological spaces. The aim of this paper is to present a common approach allowing to obtain families of intuitionistic fuzzy sets in an intuitionistic fuzzy topological space.

2. Preliminaries
Definition 2.1 [1]. Let $A$ be a nonempty fixed set. An intuitionistic fuzzy set (IFS, for short) $A$ is an object having the form $A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}$ where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote respectively the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\gamma_A(x)$) of each element $x \in X$ to the set $A$, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

Obviously, every fuzzy set $A$ on a nonempty set $X$ is an IFS having the form $A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}$.

Definition 2.2 [1]. Let $X$ be a nonempty set and let the IFS's $A$ and $B$ in the form $A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}$, $B = \{ (x, \mu_B(x), \gamma_B(x)) : x \in X \}$. Let $\{ A_j : j \in J \}$ be an arbitrary family of IFS's in $(X, \tau)$. Then,

1. $A \subseteq B$ if and only if $\forall x \in X \{ \mu_A(x) \leq \mu_B(x) \text{ and } \gamma_A(x) \geq \gamma_B(x) \}$;
2. $\overline{A} = \{ (x, \gamma_A(x), \mu_A(x)) : x \in X \}$;
3. $\cap A_j = \{ (x, \land \mu_A(x), \lor \gamma_A(x)) : x \in X \}$;
4. $\cup A_j = \{ (x, \lor \mu_A(x), \land \gamma_A(x)) : x \in X \}$;
5. $1_X = \{ (x, 1, 0) : x \in X \}$ and $0_X = \{ (x, 0, 1) : x \in X \}$.

Definition 2.3. An intuitionistic fuzzy topology [3] (IFT, for short) on a nonempty set $X$ is a family $\tau$ of IFSs in $X$ satisfying the following axioms:

(i) $0_X, 1_X \in \tau$;
(ii) $A_1 \cap A_2 \in \tau$ for every $A_1, A_2 \in \tau$;
(iii) $\cup A_j \in \tau$ for any $\{ A_j : j \in J \} \subseteq \tau$.

In this case, the ordered pair $(X, \tau)$ is called an intuitionistic fuzzy topological space (IFTS, for short) and each IFS in $\tau$ is known as an intuitionistic fuzzy open set (IFOS, for short) in $X$. The complement of an intuitionistic fuzzy open set is called an intuitionistic fuzzy closed set (IFCS, for short). The family of all IFOSs (resp. IFCSs) of $(X, \tau)$ is denoted by $\text{IFO}(X)$ (resp. $\text{IFC}(X)$).

Definition 2.4 [3]. Let $(X, \tau)$ be an IFTS and let $A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}$ be an IFS in $X$. Then the intuitionistic fuzzy interior and intuitionistic fuzzy closure of $A$ is defined by $\text{Int}(A) = \bigcup \{ G \setminus G \text{ is an IFOS in } X \text{ and } G \subseteq A \}$ and $\text{Cl}(A) = \bigcap \{ G \setminus G \text{ is an IFCS in } X \text{ and } G \supseteq A \}$.
Remark 2.5. For any IFS $A$ in $(X, \tau)$, we have, $\text{Cl}(1-A) = 1 - \text{Int}(A)$, $\text{Int}(1-A) = 1 - \text{Cl}(A)$.

Definition 2.6. An intuitionistic fuzzy point [4] (IFP, for short), written $p_{(a, b)}$, is defined to be an IFS in $X$ given by

$$p_{(a, b)} = \begin{cases}
((\alpha, \beta), & \text{if } x = p \\
(0,1), & \text{otherwise.}
\end{cases}$$

Let $X$ be a nonempty set and $\mathcal{F} = \{\lambda: X \to [0,1] \}$ be the family of all intuitionistic fuzzy sets defined on $X$. Let $\gamma: \mathcal{F} \to \mathcal{F}$ be a function such that $\lambda \leq \mu$ implies that $\gamma(\lambda) \leq \gamma(\mu)$ for every $\lambda, \mu \in \mathcal{F}$. That is, $\gamma$ is a monotonic function defined on $\mathcal{F}$ by $\Gamma(\mathcal{F})$ or simply $\Gamma$. We will defined the following subclasses of $\Gamma$.

1. For every $\alpha \in [0,1]$, define $\Gamma_{\alpha} = \{\gamma \in \Gamma \mid \gamma(\alpha) = \alpha\}$ where $\alpha$ is the intuitionistic fuzzy set defined by $\alpha(x) = \alpha$ for every $x \in X$.

2. $\Gamma_1 = \{\gamma \in \Gamma \mid \gamma^2(\lambda) = \gamma(\lambda)\}$ for every $\lambda \in \mathcal{F}$.

3. $\Gamma_\prec = \{\gamma \in \Gamma \mid \lambda \leq \gamma(\lambda)\}$ for every $\lambda \in \mathcal{F}$ and

4. $\Gamma_\succ = \{\gamma \in \Gamma \mid \gamma(\lambda) \leq \lambda\}$ for every $\lambda \in \mathcal{F}$.

If $\Sigma$ is a collection of some of the symbols $2, - , +$ and $\alpha \in [0,1]$, then $\Gamma_{\Sigma} = \{\gamma \in \Gamma \mid \gamma \in \Gamma, \text{for every } t \in \Sigma\}$.

3. Properties of $\gamma$-intuitionistic fuzzy open sets

Definition 3.1. Let $X$ be a nonempty set $\gamma \in \Gamma$. An intuitionistic fuzzy set $\lambda \in \mathcal{F}$ is said to be $\gamma$-intuitionistic fuzzy open (for short, $\gamma$-IF) if $\lambda \leq \gamma(\lambda)$.

Theorem 3.2. Let $X$ be a nonempty set, $\mathcal{F}$ be the family of all intuitionistic fuzzy sets defined on $X$ and $\gamma \in \Gamma$. Then the following hold.

1. Arbitrary union of $\gamma$-IF open sets is a $\gamma$-IF open set.

2. $\overline{T}$ is $\gamma$-IF open if and only if $\gamma \in \Gamma_1$.

3. If $\gamma \in \Gamma_2$, then every intuitionistic fuzzy set of the form $\gamma(\lambda), \lambda \mathcal{F}$ is a $\gamma$-IF open set.

4. If $\gamma \in \Gamma_\prec$, then every intuitionistic fuzzy subset is $\gamma$-IF open.

5. $\gamma \in \Gamma_\succ$, then $\lambda$ is $\gamma$-IF open if and only if $\lambda = \gamma(\lambda)$. 
(6) $\emptyset$ is always $\gamma$-IF open set.

Proof. (1) Let $\{ \lambda : i \in J \} \subseteq \mathcal{F}$. If $\lambda \leq \gamma(\lambda)$ for all $i \in J$ and $\bigvee \lambda = \lambda$, then for all $i \in J$, $\lambda \leq \gamma(\lambda) \Rightarrow \gamma(\lambda) \leq \bigvee \gamma(\lambda) \Rightarrow \lambda = \bigvee \lambda \leq \bigvee \gamma(\lambda) \leq \gamma(\lambda)$.

(2) $\bigcap_{i \in J} \lambda$ is $\gamma$-IF open if and only if $\bigcap_{i \in J} \gamma(\lambda)$ if and only if $\gamma \in \Gamma_i$.

(3) If $\gamma \in \Gamma_i$ then $\gamma(\lambda) = \gamma(\lambda)$ for every $\lambda \in \mathcal{F}$ and so $\gamma(\lambda)$ is a $\gamma$-IF open set.

(4) If $\gamma \in \Gamma_i$, then $\lambda \leq \gamma(\lambda)$ for every $\lambda \in \mathcal{F}$ is $\gamma$-IF open.

(5) Suppose $\gamma \in \Gamma_i$. If $\lambda$ is $\gamma$-IF open, then $\lambda \leq \gamma(\lambda)$ and so $\lambda = \gamma(\lambda)$. By hypothesis. If $\lambda = \gamma(\lambda)$, then clearly, $\lambda$ is $\gamma$-IF open.

(6) Clear.

Let $X$ be a nonempty set and $\mathcal{F}$ be the family of all intuitionistic fuzzy sets defined on $X$. A subfamily $\mathcal{G}$ of $\mathcal{F}$ is called generalized intuitionistic fuzzy topology if $\emptyset \in \mathcal{G}$ and $\bigvee \{ \lambda_{\alpha} : \alpha \in \Delta \} \in \mathcal{G}$ whenever $\lambda_{\alpha} \in \mathcal{G}$ for every $\alpha \in \Delta$.

Remark 3.3. If $\gamma \in \Gamma$, by Theorem 3.2 (1), it follows that $\mathcal{A}$, the family of all $\gamma$-IF open sets is a generalized intuitionistic fuzzy topology.

Definition 3.4. For $\lambda \in \mathcal{F}$, the $\gamma$-interior of $\lambda$, denote by $\text{Int}_\gamma(\lambda)$, is given by $\text{Int}_\gamma(\lambda) = \bigvee \{ \mu \in \mathcal{A} | \mu \subseteq \lambda \}$.

Proposition 3.5. Let $\gamma \in \Gamma$, $\lambda \in \mathcal{F}$ and $(X, \tau)$ be an intuitionistic fuzzy topological space. Then, for the $\gamma$-interior operator $\text{Int}_\gamma(\lambda) = \text{Int}_\gamma(\lambda)$ for all $\lambda \in \mathcal{F}$.

The following Theorem 3.6 gives all the properties of $\text{Int}_\gamma(\lambda)$.

Theorem 3.6. Let $X$ be a nonempty set, $\mathcal{F}$ the family of all intuitionistic fuzzy sets defined on $X$ and $\gamma \in \Gamma$. Then the following properties hold.

(1) For every $\lambda \in \mathcal{F}$, $\text{Int}_\gamma(\lambda)$ is the largest $\gamma$-IF open set contained in $\lambda$.

(2) $\lambda$ is $\gamma$-IF open if and only if $\text{Int}_\gamma(\lambda) = \lambda$.

(3) $\text{Int}_\gamma(\lambda) \in \mathcal{F}$ for every $\gamma \in \Gamma$.

(4) $\text{Int}_\gamma(\lambda) \in \mathcal{G}$ if and only if $\gamma \in \Gamma_i$.

(5) If $\gamma \in \lambda_{03}$, then $\gamma = t$.

(6) $\lambda$ is $\gamma$-IF open if and only if $\text{Int}_\gamma(\lambda) = \lambda$.

Proof. (1) By definition and Theorem 3.2 (1), $\text{Int}_\gamma(\lambda)$ is a $\gamma$-IF open set. By definition, $\text{Int}_\gamma(\lambda) \subseteq \lambda$. If $\beta$ is $\gamma$-IF open such that $\beta \subseteq \lambda$, then $\beta \leq \text{Int}_\gamma(\lambda) \leq \lambda$ and so $\text{Int}_\gamma(\lambda)$ is the largest $\gamma$-IF open set contained in $\lambda$. 


(2) If \( \alpha \) is \( \gamma -IF \) open, then by definition, \( \lambda \leq t_\gamma(\lambda) \) and so \( t_\gamma(\lambda) = \lambda \) by (1). The converse part is clear.

(3) Clearly, \( t_\gamma(0) = 0 \) and so \( t_\gamma \in \Gamma_0 \). By (1), \( t_\gamma(\lambda) \) is a \( \gamma -IF \) open set for every \( \lambda \in \mathcal{F} \) and so by (2), \( t_\gamma(t_\gamma(\lambda)) = t_\gamma(\lambda) \) and so \( t_\gamma \in \Gamma_2 \). By (1), it follows that \( t_\gamma \in \Gamma_+ \).

(4) \( \gamma \in \Gamma_1 \) if and only if \( \gamma(\overline{\gamma}) = \overline{\gamma} \) if and only if \( \overline{\gamma} \) is \( \gamma -IF \) open, by (2). Therefore, \( \gamma \in \Gamma_1 \) if and only if \( \gamma(\overline{\gamma}) = \overline{\gamma} \) if and only if \( \overline{\gamma} \) is \( \gamma -IF \) open, by (2). Therefore, \( \gamma \in \Gamma_1 \) if and only if \( \gamma = t_\gamma \).

(5) Let \( \lambda \in \mathcal{F} \). Since \( \gamma \in \Gamma_2 \), \( \gamma(\gamma(\lambda)) = \gamma(\lambda) \). Therefore, \( \gamma(\lambda) \) is a \( \gamma -IF \) open set. Since \( \gamma \in \Gamma_+ \), \( \gamma(\lambda) \leq \lambda \). Thus \( \gamma(\lambda) \) is a \( \gamma -IF \) open set such that \( \gamma(\lambda) \leq \lambda \). If \( \mu \leq \lambda \) is \( \gamma -IF \) open, then \( \mu \leq \gamma(\mu) \leq \gamma(\lambda) \) and so by (1), \( \gamma(\lambda) = t_\gamma(\lambda) \) and so \( \gamma = t_\gamma \).

Remark 3.7. Let \( X \) be a nonempty set and \((X, \tau)\) an intuitionistic fuzzy topological space. Let \( \text{Int} \) represent the interior operator and \( \text{Cl} \) denote the closure operator of the intuitionistic fuzzy topological space \((X, \tau)\). Then,

(1) for \( \gamma = \text{Int}, \gamma -IF \) open sets coincide with intuitionistic fuzzy open sets,

(2) for \( \gamma = \text{Int} \text{Cl}, \gamma -IF \) open sets coincide with intuitionistic fuzzy preopen sets [5],

(3) for \( \gamma = \text{Cl} \text{Int}, \gamma -IF \) open sets coincide with intuitionistic fuzzy semiopen sets [5],

(4) for \( \gamma = \text{Int} \text{Cl} \text{Int}, \gamma -IF \) open sets coincide with intuitionistic fuzzy \( \alpha \)-sets in [5],

(5) for \( \gamma = \text{Cl} \text{Int} \text{Cl}, \gamma -IF \) open sets coincide with intuitionistic fuzzy semipreopen sets in [5],

(6) for \( \gamma = \text{Int} \text{Cl} \backslash \text{Cl} \text{Int}, \gamma -IF \) open sets coincide with intuitionistic fuzzy \( \gamma \)-open sets [6].

Moreover, \( i_{\text{Int} \text{Cl}}(\lambda) \) is the intuitionistic fuzzy preinterior [6], \( i_{\text{Cl} \text{Int}}(\lambda) \) is the intuitionistic fuzzy semiinterior [6], \( i_{\text{Int} \text{Cl} \text{Int}}(\lambda) \) is the intuitionistic fuzzy semipreinterior [5], \( i_{\text{Int} \text{Cl} \text{Int}}(\lambda) \) is the intuitionistic fuzzy \( \alpha \)-interior [5] and \( i_{\text{Int} \text{Cl} \backslash \text{Cl} \text{Int}}(\lambda) \) is the intuitionistic fuzzy \( \gamma \)-interior [6] of \((\lambda) \in \mathcal{F} \).

Theorem 3.8. Let \( \gamma_1, \gamma_2 \in \Gamma \).

(1) \( \gamma_1 \gamma_2 \in \Gamma \).

(2) If \( \gamma_1, \gamma_2 \in \Gamma_n \) for all \( n \in \{0,1,+,–\} \), then \( \gamma_1 \gamma_2 \in \Gamma_n \).

Proof. One can easily prove this theorem by means of the definitions of \( \Gamma_n \) for all \( n \in \{0,1,+,–\} \).
Proposition 3.9. Let $t, \kappa \in \Gamma_2$ and for all $\lambda \in F$, $t\kappa(\lambda) \leq \kappa(t\lambda)$ (*) and $t\kappa(t\lambda) \leq \kappa(t\kappa(t\lambda))$ (**). If $\gamma$ is a composition of alternating factors $t$ and $\kappa$, then $\gamma \in \Gamma_2$ except for the case $\gamma = \kappa t$.

Proof. Let $\lambda \in F$. For $\gamma = t\kappa$, we have $t\kappa(\lambda) \leq \kappa(t\lambda)$ by (*). Hence, we obtain $t\kappa(t\lambda) \leq \kappa(t\kappa(t\lambda))$. On the other hand, by (**), we get $t\kappa(t\lambda) \leq t\kappa(t\kappa(t\lambda))$ implying that $t\kappa(t\kappa(t\lambda)) = t\kappa(t\lambda)$. Hence, we obtain $t\kappa(t\lambda) = t\kappa(t\kappa(t\lambda))$ implying that $\gamma = t\kappa(t\lambda)$. For $\gamma = t\kappa t$, we have $(t\kappa t)(t\kappa t)(\lambda) = (t\kappa t)(t\kappa t)(\lambda) = t\kappa(t\kappa(t\lambda)) = t\kappa(t\lambda)$ implying that $\gamma = t\kappa(t\lambda)$. For $\gamma = \kappa t$, we have $(\kappa t\kappa t)(\kappa t\kappa t)(\lambda) = \kappa(t\kappa t)(\kappa t\kappa t)(\lambda) = \kappa(t\kappa t)(\kappa t\kappa t)(\lambda) = \kappa(t\kappa t)(\kappa t\kappa t)(\lambda)$. Hence $\gamma = t\kappa(t\lambda)$. For $\gamma = t\kappa t\kappa$, we have $(t\kappa t\kappa)(t\kappa t\kappa)(\lambda) = t\kappa(t\kappa t\kappa)(\lambda)$. Hence $\gamma = t\kappa(t\lambda)$.

Corollary 3.11. Let $(X, \tau)$ be an intuitionistic fuzzy topological space. If $t$ and $\kappa$ are chosen as the interior and closure operators respectively, since $\text{Int}, \text{Cl} \in \Gamma_2$ and (*), (**), and (***) are satisfied by $\text{Int}$ and $\text{Cl}$ operators, any composition of alternating factors $\text{Cl}$ and $\text{Int}$ is equal to one of the mappings $\text{Int}, \text{Cl}, \text{Cl}(\text{Int}), \text{Int}(\text{Cl}), \text{Cl}(\text{Int})$, or $\text{Int}(\text{Cl}(\text{Int}))$ by Proposition 3.9 and 3.10.

Corollary 3.12. If $t \in \Gamma_2$ and $\kappa \in \Gamma_2$, then any composition of alternating factors $t$ and $\kappa$ is an element of $\Gamma_2$.

Proof. It follows from the fact that inequalities (*), (**), (***) are verified by $t$ and $\kappa$.

Proposition 3.13. Let $\gamma \in \Gamma$. Every $\gamma$-IF open set is $\gamma^n$-IF open for all $n \in N$. If $\gamma \in \Gamma_{-2}$, then $\gamma$-IF open sets coincide with $\gamma^n$-IF open sets.

Proof. Let $\gamma \in \Gamma$ and $\lambda \in F$ is a $\gamma$-IF open set. Then, $\lambda \leq \gamma(\lambda) \leq \gamma(\gamma(\lambda)) \leq \cdots \leq \gamma^{n-1}(\lambda) \leq \gamma^n(\lambda)$. Therefore, $\lambda$ is a $\gamma^n$-IF open set. For the case $\gamma \in \Gamma_{-2}$, assume
that $\lambda$ is a $\gamma^n$-IF open set. Then, $\lambda \leq \gamma^n (\lambda) \leq \gamma^{n-1} (\lambda) \leq \cdots \leq \gamma^2 (\lambda) \leq \gamma (\lambda)$. Hence, $\lambda$ is a $\gamma$-IF open set.

**Proposition 3.14.** Let $\iota, \kappa \in \Gamma_{2^-}$ and $\iota, \kappa$ satisfy the inequality (*). Then, any composition of factors $\iota$ and $\kappa$ is an element of $\Gamma_{2^-}$. Moreover, if $\gamma' \in \Gamma$ is an arbitrary composition of factors $\iota$ and $\kappa$, then

1. a $\gamma$-IF open set is an $\iota \kappa$-IF open set, where $\gamma = \gamma' \kappa$.
2. a $\gamma$-IF open set is an $\iota \kappa \iota$-IF open set, where $\gamma = \gamma' \iota$.
3. a $\gamma$-IF open set is an $\kappa \iota$-IF open set, where $\gamma = \kappa \gamma'$.
4. a $\gamma$-IF open set is an $\kappa \iota \kappa$-IF open set, where $\gamma = \kappa \gamma' \kappa$.

The converse implications hold if no factor $\iota$ is immediately followed by another factor $\iota$.

**Proof.** Let $\lambda \in \mathcal{F}$. It can be easily seen that if $\iota, \kappa \in \Gamma_{2^-}$, then $\iota^n (\lambda) \leq \iota (\lambda)$ and $\kappa^n (\lambda) \leq \kappa (\lambda)$ and by (*), we obtain $(\iota \kappa)^n (\lambda) \leq \kappa^n (\lambda) \leq \lambda$ for all $n \in \mathbb{N}$. Since for all $n \in \mathbb{N}$, $\gamma (\lambda) = \gamma' (\lambda) = \iota^n (\lambda) = \iota^{n-1} (\lambda) = \gamma (\lambda)$, we have $\gamma \in \Gamma_{2^-}$ for $\gamma = \gamma'$. Now let $\gamma = \gamma' \gamma_2$, where $\gamma' \gamma_2$ are any (may be empty) compositions of factors $\iota$ and $\kappa$ that is, $\gamma$ has at least one factor $\kappa$. Then, $\gamma (\lambda) = \gamma' \gamma_2 \gamma' \gamma_2 (\lambda)$. If $\iota$ is replaced instead of each $\iota^n$ factors, and $\kappa$ is replaced instead of each $\kappa^n$ factors, and $\kappa$ is replaced instead of each $(\iota \kappa)^n$ factors in the composition $\gamma' \gamma_2 \gamma' \gamma_2$, we get $\gamma (\lambda) = \gamma' \gamma_2 \gamma' \gamma_2 (\lambda) \leq \gamma' \gamma_2 (\lambda) = \gamma (\lambda)$. Hence $\gamma \in \Gamma_{2^-}$.

1. Let $\gamma = \gamma' \kappa$ and $\lambda \in \mathcal{F}$ is a $\gamma$-IF open set. For a suitable $m \in \mathbb{N}$, $\lambda \leq \gamma (\lambda) = \gamma' \kappa (\lambda) \leq (\iota \kappa)^m \lambda = (\iota \kappa)^m (\lambda) \leq \kappa (\lambda) = \kappa (\lambda)$ by similar substitutions in the above manner. Hence $\lambda$ is an $\iota \kappa$-IF open set. Conversely, suppose that no factor $\iota$ is immediately followed by another factor $\iota$ and $\lambda$ is $\iota \kappa$-IF open set. We have $\lambda \leq \kappa (\lambda)$ implying that $\lambda \leq (\iota \kappa)^n (\lambda)$, where $n$ is the number of the factors $\kappa$ in $\gamma$ and also we get $\lambda \leq \iota \kappa (\lambda) \leq \kappa (\lambda)$. Then by the inequality (*) and $\lambda \leq \kappa (\lambda)$, we obtain $\lambda \leq \gamma (\lambda)$. Hence, $\lambda$ is a $\gamma$-IF open set.

2. Let $\gamma = \gamma' \iota$ and $\lambda$ is a $\gamma$-IF open set. By (*), we obtain $\lambda \leq \gamma' \iota (\lambda) \leq (\iota \kappa)^m \iota (\lambda) \leq (\iota \kappa)^m \iota (\lambda) \leq \kappa (\lambda) = \kappa (\lambda)$ for a suitable $m \in \mathbb{N}$. Hence, $\lambda$ is an $\iota \kappa$-IF open set. Conversely, if no factor $\iota$ is immediately followed by another factor $\iota$ and $\lambda$ is $\iota \kappa$-IF open set, then for $m \in \mathbb{N}$, $\lambda \leq \iota \kappa (\lambda) \Rightarrow (\iota \kappa)^m \iota (\lambda) \leq (\iota \kappa)^m \iota \kappa (\lambda) \leq (\iota \kappa)^m \iota (\lambda) = \gamma (\lambda)$. Therefore, by the inequality (*), $\lambda \leq \iota \kappa (\lambda) \leq (\iota \kappa)^m \iota (\lambda) \leq \gamma (\lambda)$, where $m$ is the number of the factors $\kappa$ in $\gamma$. Thus $\lambda$ is a $\gamma$-IF open set.
(3) Let $\gamma = \kappa \gamma' t$ and $\lambda$ is a $\gamma$-$IF$ open set. For $m \geq 2$ and $m \in N$, we have $\lambda \subseteq \gamma(\lambda)$ $\subseteq (\kappa^m \lambda) = \kappa (\kappa^{m-1} t (\lambda)) \subseteq \kappa \lambda$. Hence $\lambda$ is a $\kappa IF$ open set. Conversely, if no factor $t$ is immediately followed by another factor $t$ and $\lambda$ is $\kappa IF$ open set, then by the inequality (*), we have $\lambda \subseteq \kappa (\lambda) \Rightarrow \lambda \subseteq (\kappa^m (\lambda) \subseteq \gamma (\lambda)$, where $n$ is the number of factors $\kappa$ in $\gamma$. Hence $\lambda$ is a $\gamma$-$IF$ open set.

(4) Let $\gamma = \kappa \gamma' k$ and $\lambda$ is a $\gamma$-$IF$ open set. For a suitable $m \in N$, we have $\lambda \subseteq \kappa \gamma' k(\lambda) \subseteq (\kappa^m \lambda) = \kappa (\kappa^{m-1} \gamma' \kappa (\lambda) \subseteq \kappa \kappa \kappa (\lambda).$ Thus, $\lambda$ is a $\kappa \kappa \kappa$-$IF$ open set. Conversely, if no factor $t$ is immediately followed by another factor $t$ and $\lambda$ is $\kappa \kappa \kappa$-$IF$ open set, then by the inequality (*), we obtain $\lambda \subseteq \kappa \kappa \kappa (\lambda) \subseteq (\kappa^m (\lambda) \subseteq \gamma (\lambda)$, for all $m \in N$. Thus, by the inequality (*), we obtain $\lambda \subseteq \gamma \kappa (\lambda) \subseteq (\kappa^m (\lambda) \subseteq \gamma (\lambda)$ where $n$ is the number of factors $\kappa$ in $\gamma$. Hence, $\lambda$ is a $\gamma$-$IF$ open set.

**Corollary 3.15.** Let $t \in \Gamma_i$ and $k \in \Gamma_0$. Then, the statements of Proposition 3.14 are valid. Furthermore

(1) Every $1k\kappa$-$IF$ open set is $1k\kappa$-$IF$ open set.
(2) Every $1k\kappa$-$IF$ open set is $k\kappa$-$IF$ open set.
(3) Every $1k\kappa$-$IF$ open set is $k\kappa\kappa$-$IF$ open set.
(4) Every $k\kappa$-$IF$ open set is $k\kappa\kappa$-$IF$ open set.
(5) Every $k\kappa\kappa$-$IF$ open set is $k$-$IF$ open set.
(6) Every $1k\kappa$-$IF$ open set is $1k\kappa$-$IF$ open set.
(7) If $\lambda \in \mathcal{F}$ is an $1k\kappa$-$IF$ open set and $k\kappa$-$IF$ open set, then it is $1k\kappa$-$IF$ open set.

**Proof.** Let $\lambda \in \mathcal{F}$. It is clear from the fact that the inequality (*) and $t \in \Gamma_0$ are satisfied under these conditions.

(1) Let $\lambda$ is an $1k\kappa$-$IF$ open set. Hence, $\lambda \subseteq t(\kappa t)(\lambda) \subseteq k\kappa (\lambda)$ implying that $\lambda$ is a $k\kappa IF$ open set. If $\lambda \subseteq t(\kappa t)(\lambda) \subseteq k\kappa (\lambda)$, then $\lambda$ is a $k\kappa$-$IF$ open set. Providing that $\lambda$ is a $k\kappa$-$IF$ open set or $k\kappa$-open set, then $\lambda \subseteq k\kappa (\lambda) \subseteq k\kappa k(\lambda)$ or $\lambda \subseteq k\kappa (\lambda) \subseteq k\kappa k(\lambda)$, respectively. That is, $\lambda$ is a $k\kappa k$-$IF$ open set in both cases. For the case $\lambda \subseteq k\kappa k(\lambda)$, we get $\lambda \subseteq k\kappa k(\lambda) \subseteq k\kappa (\lambda) \subseteq k\kappa (\lambda)$ implying that $\lambda$ is a $k\kappa$-$IF$ open set.

(2) Suppose that $\lambda$ is an $k\kappa$-$IF$ open set and $k\kappa$-open set. Hence, we obtain $\lambda \subseteq k\kappa (\lambda) \subseteq 1k\kappa (\lambda) \subseteq 1k\kappa (\lambda)$. Thus, $\lambda$ is an $1k\kappa$-$IF$ open set.

**Remark 3.16.** If $t$ and $\kappa$ are choosen as the interior and closure operator, respectively, and Remark 3.7 considered, then we have the following

(1) An $1k\kappa$-$IF$ open set need not be an $1k\kappa$-$IF$ open set,
(2) A $k\kappa$-$IF$ open set need not be an $1k\kappa$-$IF$ open set,
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(3) A $\kappa\kappa$-IF open set need not be a $\kappa$-IF open set,
(4) A $\kappa\kappa$-IF open set need not be an $\kappa$-IF open set,
(5) An intuitionistic fuzzy set need not be a $\kappa\kappa$-IF open set,
(6) The $\kappa$-IF open sets and $\kappa$-open sets are independent notions.

Let $(X, \tau)$ be an intuitionistic fuzzy topological space. We denote the family of all $\gamma \in \Gamma(X)$ satisfying the property $\forall \in \tau, \forall \lambda \in F; \mu \land \gamma(\lambda) \leq \gamma(\mu \land \lambda)$ by $\Gamma_3$. It is clear that for an intuitionistic fuzzy topological space $(X, \tau)$, the interior operator $\text{Int} \in \Gamma_3$. On the other hand, a closure operator may not be an element of $\Gamma_3$.

**Proposition 3.17.** If $\gamma_1, \gamma_2 \in \Gamma_3$, then $\gamma_1 \circ \gamma_2 \in \Gamma_3$.

**Proof.** Let $(X, \tau)$ be an intuitionistic fuzzy topological space, $\gamma_1, \gamma_2 \in \Gamma_3$, $\mu \in \tau$, and $\lambda \in F$. Since $\mu \land \gamma_1 \circ \gamma_2(\lambda) = \mu \land \gamma_1(\gamma_2(\lambda)) \leq \gamma_1(\mu \land \gamma_2(\lambda)) \leq \gamma_1(\gamma_2(\mu \land \lambda)) = \gamma_1 \circ \gamma_2(\mu \land \lambda)$, we have $\gamma_1 \circ \gamma_2 \in \Gamma_3$.

**Proposition 3.18.** Let $(X, \tau)$ be an intuitionistic fuzzy topological space and $\gamma \in \Gamma_3$. If $\mu \in \tau$, and $\lambda \in F$ is a $\gamma$-IF open set, then $\mu \land \lambda$ is a $\gamma$-IF open set.

**Proof.** Since $\mu \leq \mu$ and $\lambda \leq \gamma(\lambda)$, we have $\mu \land \lambda \leq \mu \land \gamma(\lambda) \leq \gamma(\mu \land \lambda)$. Hence $\mu \land \lambda$ is a $\gamma$-IF open set.

**Proposition 3.19.** Let $(X, \tau)$ be an intuitionistic fuzzy topological space. If $\gamma \in \Gamma_3$ and $\mu$ is an intuitionistic fuzzy open set such that $\mu \leq \gamma(\overline{1})$, then $\mu$ is a $\gamma$-IF open set.

**Proof.** Let $\mu \in \tau$ and $\mu \leq \gamma(\overline{1})$. Since $\mu = \mu \land \gamma(\overline{1}) \leq \gamma(\mu \land \overline{1}) = \gamma(\mu)$, $\mu$ is a $\gamma$-IF open set.

**Corollary 3.20.** Let $\gamma \in \Gamma_{13}$. Then every intuitionistic fuzzy open set is a $\gamma$-IF open set.

**Proof.** Let $\mu \in \tau$. Then, $\mu = \mu \land \gamma(\overline{1}) = \mu \land \gamma(\overline{1}) \leq \gamma(\mu \land \overline{1}) = \gamma(\mu)$. Hence $\mu$ is a $\gamma$-IF open set.

**Corollary 3.21.** Let $(X, \tau)$ be an intuitionistic fuzzy topological space. If $\gamma \in \Gamma_3$ and $\mu$ is an intuitionistic fuzzy open set such that there exists a $\gamma$-IF open set $\lambda$ containing $\mu$, then $\mu$ is a $\gamma$-IF open set.

**Proof.** Let $\lambda \in F$, $\lambda \leq \gamma(\lambda)$ and $\mu \leq \lambda$. Since $\mu \leq \lambda \leq \gamma(\lambda) \leq \gamma(\overline{1})$, $\mu$ is a $\gamma$-IF open set from Proposition 3.19.
Proposition 3.22. If $\gamma \in \Gamma_3$, then $i_{\gamma} \in \Gamma_3$.

Proof. Let $\mu \in \tau$ and $\lambda \in \mathcal{F}$. Since $\mu \land i_{\gamma}(\lambda)$ is a $\gamma$-$IF$ open set by Proposition 3.18 and $\mu \land i_{\gamma}(\lambda) \leq \mu \land \lambda$, we have $\mu \land i_{\gamma}(\lambda) \leq i_{\gamma}(\mu \land f)$. Hence, we have $i_{\gamma} \in \Gamma_3$.

Proposition 3.23. Let $\gamma \in \Gamma_1$ if and only if $\text{Int}(\lambda) \leq \gamma(\lambda)$ for all $\lambda \in \mathcal{F}$.

Proof. Let $\lambda \in \mathcal{F}$. If $\gamma \in \Gamma_3$, we have $\text{Int}(\lambda) \land \gamma(\lambda) = \text{Int}(\lambda) \land \overline{\lambda} \leq \gamma(\text{Int}(\lambda)) \leq \gamma(\lambda)$. Since $\text{Int}(\overline{\lambda}) \leq \gamma(\overline{\lambda}) \leq \overline{\lambda}$, we get $\gamma \in \Gamma_1$.

Remark 3.24. Consider the function $\gamma'$ consisting of alternating compositions of $\text{Int}$ and $\gamma$ where $\gamma \in \Gamma_{23}$. It is clear that $\gamma' \in \Gamma_3$ by Proposition 3.17.

Proposition 3.25. Let $\gamma \in \Gamma_{23}$ and $\gamma'$ consist of alternating compositions of $\text{Int}$ and $\gamma$. Then $\gamma' \in \Gamma_3$ except for the condition $\gamma' = \gamma \text{Int}$.

Proof. Since the inequalities (*) and (**) are satisfied for the cases $t = \text{Int}$ and $\kappa = \gamma$, we have $\gamma' \in \Gamma_3$ by Proposition 3.9.

Remark 3.26. If $\gamma \in \Gamma_{23}$, then the case $\text{Int}$, $\gamma$, $\text{Int}\gamma$, $\gamma\text{Int}$, $\gamma\text{Int}\gamma$, $\gamma'\text{Int}\gamma$ are enough to be considered.

Proposition 3.27. If $\gamma \in \Gamma_{23}$, then

1. $\text{Int}(\gamma) \leq \gamma \text{Int}$.
2. $\text{Int}(\gamma) \leq \gamma\text{Int}(\gamma)$.
3. $\gamma'\text{Int} \leq \gamma$.
4. $\gamma'\text{Int} \leq \gamma$.

Proof. (3) and (4) are clear from the fact that $\gamma \in \Gamma_3$ and $\text{Int} \in \Gamma_\omega$. We need to show that $\text{Int}(\gamma)(\lambda) \leq \gamma\text{Int}(\gamma)(\lambda)$ for all $\lambda \in \mathcal{F}$. Since $\text{Int}(\gamma)(\lambda) \leq \gamma(\text{Int}(\lambda)) \leq \gamma(\lambda)$, $\gamma(\lambda)$ is a $\gamma$-$IF$ open set by Proposition 3.19. Hence, we have $\text{Int}(\gamma)(\lambda) \leq \gamma\text{Int}(\lambda)$ which completes for the proofs of (1) and (2).

Proposition 3.28. Let $(X, \tau)$ be an intuitionistic fuzzy topological space. If $\gamma \in \Gamma_{123}$, then for all $\lambda \in \mathcal{F}$, $\text{Int}(\lambda) \leq \gamma \text{Int}(\lambda) \leq \gamma\text{Int}(\lambda) \leq \gamma\text{Int}(\lambda) \leq \gamma(\lambda)$.

Proof. Let $\lambda \in \mathcal{F}$. If $\gamma \in \Gamma_{123}$, then by Proposition 3.19, we have $\text{Int}(\lambda) \leq \gamma\text{Int}(\lambda)$. Since the inequalities (*) and (**) are satisfied for the case $\kappa = \gamma$ and $t = \text{Int}$, we get $\gamma\text{Int} \in \Gamma_2$. Thus we obtain $\text{Int}(\lambda) = \text{Int}(\lambda) \leq \gamma\text{Int}(\lambda) \leq \gamma\text{Int}(\lambda) = \gamma\text{Int}(\lambda) \leq \gamma(\lambda) = \gamma(\lambda)$.
Corollary 3.29. If $\gamma \in \Gamma_{123}$, then the implications in the following diagram hold:

\[
\begin{array}{c}
\text{Int } \gamma{-}\text{IF open} \\
\text{open } \rightarrow \text{Int} \gamma\text{-IF open} \\
\gamma\text{-IF open} \Rightarrow \gamma\text{-IF open}
\end{array}
\]

Theorem 3.30. Let $\gamma \in \Gamma_{123}$. Then, for $n \in N$, $\lambda \in \mathcal{F}$ is

1. $(\text{Int}\gamma^n)^{-}\text{IF open} \Leftrightarrow \text{Int}\gamma\text{-IF open}$.
2. $(\text{Int}\gamma^n)\text{-IF open} \Leftrightarrow \text{Int}\gamma\text{-IF open}$.
3. $(\gamma\text{Int})^n\text{-IF open} \Leftrightarrow \gamma\text{Int}\text{-IF open}$.
4. $(\gamma\text{Int})^n\gamma\text{-IF open} \Leftrightarrow \gamma\text{Int}\gamma\text{-IF open}$.
5. Int$\gamma\text{Int}\gamma\text{-IF open} \Leftrightarrow \text{Int}\gamma$ and $\gamma\text{Int}\text{-IF open}$.
6. $\{\text{Int}\gamma\text{-IF open}, \gamma\text{Int}\gamma\text{-IF open}\} \Rightarrow \gamma\text{Int}\gamma\text{-IF open} \Rightarrow \gamma\text{-IF open}$.

Proof. (1), (2), (3) and (4) are clear from Proposition 3.13. (5) and (6) are obvious from Corollary 3.15.

Theorem 3.31. Let $(X, \tau)$ be an intuitionistic fuzzy topological space and $\gamma \in \Gamma_{13}$. Then the family $\mathcal{O}$ of all $\gamma$-IF open sets satisfies the following properties:

1. If $\lambda \in \tau$, then $\lambda \in \mathcal{O}$.
2. If $\lambda, \mu \in \mathcal{O}$ for all $i \in J$, then $\bigvee_{i \in J} f_i \in \mathcal{O}$.
3. If $\mu \in \tau$ and $\lambda \in \mathcal{O}$, then $\lambda \wedge f \in \mathcal{O}$.

Conversely, if $\mathcal{O}$ is a subset of an intuitionistic fuzzy topological spaces satisfying the properties from (1) to (3), then there exists a function $\gamma \in \Gamma_{0123}$ such that $\mathcal{O}$ consists of all $\gamma$-IF open sets.

Proof. (1), (2) and (3) are obvious from Corollary 3.20, Proposition 3.2 (1) and 3.18, respectively. Conversely, let $\mathcal{O}$ be a subset of an intuitionistic fuzzy topological space satisfying the properties from (1) to (3). Define $\gamma : \mathcal{F} \rightarrow \mathcal{F}$. $\lambda \rightarrow \gamma(\lambda) = \bigvee \{ \mu \in \mathcal{O} \mid \mu \leq f \}$. It is clear to see that $\gamma \in \Gamma_{123}$. Let $\mu \in \mathcal{O}$ and $\lambda \in \mathcal{F}$. Since $\mu \wedge \gamma(\lambda) \in \mathcal{O}$ by (3) and $\mu \wedge \gamma(\lambda) \leq \mu \wedge \lambda$, we have $\mu \wedge \gamma(\lambda) \leq \gamma(\mu \wedge \lambda)$. Hence, $\gamma \in \Gamma_{3}$. Now it is enough to show that $\mathcal{O}$ consists of all $\gamma$-IF open sets. Let $\lambda \in \mathcal{O}$. Then we have $\gamma(\lambda) = \lambda$ implying that $\lambda$ is a $\gamma$-IF open set. Conversely, $\lambda \leq \gamma(\lambda)$, then $\lambda = \gamma(\lambda) \in \mathcal{O}$. 

Definition 3.32. Let $X$ be a nonempty set, $\mathcal{F}$ be the family of all intuitionistic fuzzy sets defined on $X$ and $\gamma \in \Gamma$. An intuitionistic fuzzy subset $\lambda \in \mathcal{F}$ is said to be a $\gamma$-$IF$ closed set if $\overline{\gamma - \lambda}$ is a $\gamma$-$IF$ open set.

Definition 3.33. The intersection of all $\gamma$-$IF$ closed sets containing $\lambda \in \mathcal{F}$ is called the $\gamma$-closure of $\lambda$ and is denoted by $c_\gamma (\lambda)$.

The following Theorem 3.34 gives some properties of $\gamma$-$IF$ closed set and the $\gamma$-closure operator. The easy proof of the theorem is omitted.

Theorem 3.34. Let $X$ be a nonempty set, $\mathcal{F}$ be the family of all intuitionistic fuzzy sets defined on $X$ and $\gamma \in \Gamma$. Then the following hold.

1. $\overline{1}$ is a $\gamma$-$IF$ closed set.
2. Arbitrary intersection of $\gamma$-$IF$ closed sets is a $\gamma$-$IF$ closed set.
3. $\overline{0}$ is $\gamma$-$IF$ closed if and only if $\gamma \in \Gamma_1$.
4. If $\gamma \in \Gamma_+$, then every $\lambda \in \mathcal{F}$ is $\gamma$-$IF$ closed.
5. For every $\lambda \in \mathcal{F}$, $c_\gamma (\lambda)$ is the smallest $\gamma$-$IF$ closed subset containing $\lambda$.
6. $\lambda$ is $\gamma$-$IF$ closed if and only if $c_\gamma (\lambda) = \lambda$.
7. $c_\gamma \in \Gamma_{12}$, for every $\gamma \in \Gamma$.
8. $c_\gamma \in \Gamma_0$ if and only if $\gamma \in \Gamma_1$.
9. $\lambda$ is $\Gamma_0$-$IF$ closed if and only if $c_\gamma (\lambda) = \lambda$.

Let $X$ be a nonempty set, $\mathcal{F}$ be the family of all intuitionistic fuzzy sets defined on $X$ and $\gamma \in \Gamma$. Define $\gamma^* : \mathcal{F} \to \mathcal{F}$ by $\gamma^* (\lambda) = \overline{\gamma (\overline{\lambda})}$ for every $\lambda \in \mathcal{F}$. The following Theorem 3.35 gives properties of $\gamma^*$.

Theorem 3.35. (1) $\gamma^* \in \Gamma$.

(2) $\left( \gamma^* \right)^* = \gamma$.

(3) $\gamma \in \Gamma_0$ if and only if $\gamma^* \in \Gamma_1$.

(4) $\gamma \in \Gamma_1$ if and only if $\gamma^* \in \Gamma_0$.

(5) $\gamma \in \Gamma_2$ if and only if $\gamma^* \in \Gamma_2$.

(6) $\gamma \in \Gamma_+$ if and only if $\gamma^* \in \Gamma_-$.

(7) $\left( t_\gamma \right)^* = c_\gamma$.

(8) $\left( c_\gamma \right)^* = t_\gamma$.

(9) $t_\gamma (\overline{\lambda}) = \overline{\gamma (\overline{\lambda})}$ for every $\lambda \in \mathcal{F}$.
(10) \( c_\gamma (\overline{T-\lambda}) = \overline{T-t_\gamma (\lambda)} \) for every \( \lambda \in \mathcal{F} \)

Proof. (1) Let \( \lambda, \mu \in \mathcal{F} \), such that \( \lambda \leq \mu \). Then \( \overline{T-\gamma (\overline{T-\lambda})} \leq \overline{T-\gamma (\overline{T-\mu})} \) and so \( \gamma' (\lambda) \leq \gamma' (\mu) \). Hence \( \gamma' \in \Gamma \).

(2) For \( \lambda \in \mathcal{F}, \ (\gamma')^+ (\lambda) = \overline{T-\gamma' (\overline{T-\gamma (\overline{T-\lambda})})} = \gamma (\lambda) \).

(3) \( \gamma' \in \Gamma_1 \) if and only if \( \gamma (\overline{0}) = \overline{0} \) if and only if \( \gamma \in \Gamma_0 \).

(4) The proof follows from (3).

(5) For \( \lambda \in \mathcal{F}, \ \gamma^* \in \Gamma_2 \) if and only if \( \gamma (\overline{T-\lambda}) \leq \overline{T-\lambda} \) if and only if \( \overline{T-\gamma (\overline{T-\gamma (\overline{T-\lambda})})} = \overline{T-\gamma (\lambda)} \) if and only if \( \overline{T-\gamma (\gamma (\lambda))} = \overline{T-\gamma (\lambda)} \) and so \( \gamma (\gamma (\lambda)) = \gamma (\lambda) \) if and only if \( \gamma \in \Gamma_2 \).

(6) For \( \lambda \in \mathcal{F}, \ \gamma^* \in \Gamma_\infty \) if and only if \( \gamma (\overline{T-\lambda}) \leq \overline{T-\lambda} \) if and only if \( \overline{T-\gamma (\lambda)} \leq \overline{T-\lambda} \) if and only if \( \lambda \leq \gamma (\lambda) \) if and only if \( \gamma \in \Gamma_+ \).

(7) \( t_\gamma (\overline{T-\lambda}) \), is the largest \( \gamma\text{-IF} \) open set contained in \( \overline{T-\lambda} \). Hence \( \overline{T-t_\gamma (\overline{T-\lambda})} \), is the smallest \( \gamma\text{-IF} \) closed set containing \( \lambda \) and so \( \overline{T-t_\gamma (\overline{T-\lambda})} = t_\gamma (\lambda) = c_\gamma (\lambda) \). Hence \( c_\gamma = t_\gamma^+ \).

(8) The proof follows from (7).

(9) For \( \lambda \in \mathcal{F}, \) from (8), \( t_\gamma (\overline{T-\lambda}) = c_\gamma (\overline{T-\lambda}) = \overline{T-c_\gamma (\lambda)} \).

(10) The proof follows from (9).

By Theorem 3.35 (1) above, \( \gamma' \in \Gamma \) and so we can define the collection of all \( \gamma'\text{-IF} \) open sets and \( \gamma'\text{-IF} \) closed sets. Hence, we can define the \( \gamma' \)-closure for a \( \lambda \in \mathcal{F} \).

The following Theorem 3.36 gives a characterization of \( \gamma'\text{-IF} \) closed sets and Theorem 3.37 below gives a property of \( \gamma' \)-closure operator \( c_\gamma \).

**Theorem 3.36.** Let \( X \) be a nonempty set, \( \mathcal{F} \) be the family of all intuitionistic fuzzy sets defined on \( X \) and \( \gamma \in \Gamma \). Then \( \lambda \) is a \( \gamma'\text{-IF} \) closed set if and only if \( \gamma (\lambda) \leq \lambda \).

Proof. \( \lambda \) is a \( \gamma'\text{-IF} \) closed set if and only if \( \overline{T-\lambda} \) is a \( \gamma'\text{-IF} \) open set if and only if \( \overline{T-\lambda} \leq \overline{T-\gamma (\overline{T-\gamma (\overline{T-\lambda})})} \) if and only if \( \overline{T-\lambda} \leq \overline{T-\gamma (\lambda)} \) if and only if \( \gamma (\lambda) \leq \lambda \).

**Theorem 3.37.** Let \( X \) be a nonempty set, \( \mathcal{F} \) be the family of all intuitionistic fuzzy sets defined on \( X \) and \( \gamma \in \Gamma_{12} \). Then \( \gamma = c_\gamma \).

Proof. Let \( \lambda \in \mathcal{F} \). Since \( \gamma \in \Gamma_2, \ \gamma' (\gamma (\lambda)) = \gamma (\lambda) \). Therefore by Theorem 3.36, \( \gamma (\lambda) \) is a \( \gamma'\text{-IF} \) closed set. Since \( \gamma \in \Gamma_+, \lambda \leq \gamma (\lambda) \). Thus \( \gamma (\lambda) \) is a \( \gamma'\text{-IF} \) closed set such
that \( \lambda \leq \gamma \lambda \). If \( \mu \) is \( \gamma \)-IF closed set such that \( \lambda \leq \mu \), then \( \lambda \leq \gamma \lambda \leq \gamma \mu \) and so by Theorem 3.34 (5), \( \gamma \lambda = c_{\gamma \lambda} (\lambda) \) and so \( \gamma = c_{\gamma \lambda} \).

**Theorem 3.38.** Let \( X \) be a nonempty set, \( \mathcal{F} \) the family of all intuitionistic fuzzy sets defined on \( X \) and \( \gamma_1, \gamma_2 \in \Gamma \). Then the following hold.

1. \( \gamma_1, \gamma_2 \in \Gamma \).
2. If \( \gamma_1, \gamma_2 \in \Gamma, \) for \( \lambda \in \sigma - \{2\} \), then \( \gamma_1 \gamma_2 \in \Gamma \).
3. \( \gamma_1 \gamma_2 = \gamma_1 \gamma_2 \).

**Proof.** (1) Let \( \lambda, \mu \in \mathcal{F} \) such that \( \lambda \leq \mu \). Since \( \gamma_2 \in \Gamma \), \( \gamma_1 \gamma_2 \leq \gamma_1 (\mu) \). Therefore, \( \gamma_1 \gamma_2 \in \Gamma \).

(2) The proof is clear.

(3) For \( \lambda \in \mathcal{F} \), \( \gamma_1 \gamma_2 \gamma_1 \gamma_2 = (\lambda - \gamma_1 \gamma_2 (\lambda - \gamma_1 \gamma_2 (\lambda)) = \lambda - \gamma_1 (\lambda - \gamma_1 (\lambda - \gamma_1 (\lambda))) = \lambda - \gamma_1 (\lambda - \gamma_1 (\lambda)) = \gamma_1 (\lambda) \).

**Theorem 3.39.** Let \( X \) be a nonempty set, \( \mathcal{F} \) the family of all intuitionistic fuzzy sets defined on \( X \) and \( i, k \in \Gamma \). Suppose \( i \cdot k (\lambda) \leq k (\lambda) \) and \( i \cdot k (\lambda) \leq k (\lambda) \) for every \( \lambda \in \mathcal{F} \). If \( \gamma \) is a product of alternating factors \( i \) and \( k \), except \( \gamma = k i \), then \( \gamma \in \Gamma \).

In addition, if \( i (\lambda) \leq k (\lambda) \) for every \( \lambda \in \mathcal{F} \) and \( \gamma = k i \), then \( \gamma \in \Gamma \).

**Proof.** \( i, k \in \Gamma \) implies that \( i, k \in \Gamma \) and so by Theorem 3.38 (1), \( \gamma \in \Gamma \). Let \( \lambda \in \mathcal{F} \). By hypothesis \( i \cdot k (\lambda) \leq k (\lambda) \) and so \( k (\lambda) \). Therefore, \( i \cdot k (\lambda) \leq k (\lambda) \). Again by hypothesis \( i (\lambda) \leq k (\lambda) \) and so \( i (\lambda) \). Hence \( i \cdot k (\lambda) \leq k (\lambda) \) and so \( i \cdot k (\lambda) \leq k (\lambda) \). Further, \( (i \cdot k) (i \cdot k) = i (i \cdot k) = i (k \cdot i) = k (i \cdot k) = i (k \cdot i) = i (k \cdot i) \).

Thus the statement is valid for three or four factors. Since \( i \cdot k \leq k i \) and so \( i \cdot k \leq k i \) and so \( i \cdot k \leq k i \) and so \( i \cdot k \leq k i \) and so \( i \cdot k \leq k i \) and so \( i \cdot k \leq k i \) and so \( i \cdot k \leq k i \). Hence it follows that \( i \cdot k \leq k i \) and so \( i \cdot k \leq k i \).

**Corollary 3.40.** Let \( X \) be a nonempty set, \( \mathcal{F} \) the family of all intuitionistic fuzzy sets defined on \( X \), \( i \in \Gamma \) and \( k \in \Gamma \). If \( \gamma \) is a product of factors of \( i \) and \( k \) then \( \gamma \in \Gamma \).

4. On \( \gamma \)-intuitionistic fuzzy semiopen sets
In this section, we introduce a new class of fuzzy set called $\gamma$-intuitionistic fuzzy semioprn sets, which contains the class of all $\gamma$-IF open sets. We characterize such sets and establish some of its properties.

**Definition 4.1.** Let $X$ be a nonempty set, $\mathcal{F}$ the family of all fuzzy sets defined on $X$ and $\gamma \in \Gamma$. $\lambda \in \mathcal{F}$ is said to be a $\gamma$-IF semiopen (briefly, $\gamma$-IF semioprn) set if there exists a $\gamma$-IF open set $\mu$ such that $\mu \leq \lambda \leq c_{\gamma}(\mu)$.

We will denote the family of all $\gamma$-IF semiopen set by $\mathcal{S}_{\gamma}(\lambda)$. The following Theorem 4.2 gives some properties of $\gamma$-IF semiopen sets.

**Theorem 4.2.** Let $X$ be a nonempty set, $\mathcal{F}$ the family of all intuitionistic fuzzy sets defined on $X$ and $\gamma \in \Gamma$. Then the following hold.

1. Every $\gamma$-IF open set is a $\gamma$-IF semiopen set.
2. $0$ is a $\gamma$-IF semiopen set.
3. Arbitrary union of $\gamma$-IF semiopen sets is a $\gamma$-IF semiopen set.

**Proof.** (1) Suppose $\lambda$ is $\gamma$-IF open. Then, $\lambda = \lambda \leq c_{\gamma}(\lambda)$ and so $\lambda$ is $\gamma$-IF semiopen.

(2) Let $\{\lambda_\alpha | \alpha \in \Delta\}$ be the family of $\gamma$-IF semiopen sets. Let $\lambda = \bigvee \lambda_\alpha$. Since each $\lambda_\alpha$ is $\gamma$-IF semiopen, there exist a $\gamma$-IF open set $\mu_\alpha$ such that $\mu_\alpha \leq \lambda_\alpha \leq c_{\gamma}(\mu_\alpha)$ and so $\bigvee \mu_\alpha \leq \bigvee \lambda_\alpha \leq \bigvee c_{\gamma}(\mu_\alpha)$. Let $\mu = \bigvee \mu_\alpha$. Then $\mu$ is a $\gamma$-IF open set. Also, $\bigvee c_{\gamma}(\mu_\alpha) \leq \bigvee c_{\gamma}(\mu) \leq \bigvee c_{\gamma}(\mu_\alpha) = c_{\gamma}(\mu)$. Thus $\mu$ is a $\gamma$-IF open set such that $\mu \leq \bigvee \lambda_\alpha \leq c_{\gamma}(\mu)$ and so $\bigvee \mu_\alpha$ is a $\gamma$-IF semiopen set.

**Theorem 4.3.** Let $X$ be a nonempty set, $\lambda \in \mathcal{F}$ and $\gamma \in \Gamma$. Then the following hold.

1. $c_{\sigma}(\lambda)$ is the smallest $\gamma$-IF semiclosed set containing $\lambda$.
2. $\lambda$ is $\gamma$-IF semiclosed if and only if $\lambda = c_{\sigma}(\lambda)$.
3. $c_{\sigma} \in \Gamma_{\text{coz}} \ (c_{\sigma} \in \Gamma_0 \text{ by Theorem 4.2 (b) and } c_{\sigma} \in \Gamma_1 \text{ by Theorem 4.2 (3))}.$
4. $t_{\sigma}(\lambda)$ is the largest $\gamma$-IF semiopen set contained in $\lambda$.
5. $\lambda$ is $\gamma$-IF semiopen if and only if $\lambda = t_{\sigma}(\lambda)$.
6. $t_{\sigma} \in \Gamma_{\text{coz}}$.

7. If $x_{(\alpha, \beta)}$ is an intuitionistic fuzzy point, then $x_{(\alpha, \beta)} \in t_{\sigma}(\lambda)$ if and only if there is a $\gamma$-IF semiopen set $\mu$ containing $x_{(\alpha, \beta)}$ such that $x_{(\alpha, \beta)} \in \mu \leq \lambda$.

The following Theorem 4.4 gives the relation between $t_{\sigma}$ and $c_{\sigma}$.

**Theorem 4.4.** Let $X$ be a nonempty set and $\gamma \in \Gamma$. Then the following hold.
\[
\begin{align*}
(1) \; t'_\sigma &= c_\sigma. \\
(2) \; c'_\sigma &= t_\sigma. \\
(3) \; t_\sigma (\overline{1} - \lambda) &= \overline{1} - c_\sigma (\lambda) \text{ for every } \lambda \in F. \\
(4) \; c_\sigma (\overline{1} - \lambda) &= \overline{1} - t_\sigma (\lambda) \text{ for every } \lambda \in F.
\end{align*}
\]

Proof. (1) Let \( \lambda \in F \). Then \((t_\sigma)' (\lambda) = \overline{1} - t_\sigma (\overline{1} - \lambda)\). Since \( t_\sigma (\overline{1} - \lambda) \) is the largest \( \gamma \)-\(IF\) semiopen set contained in \( \overline{1} - \lambda \), \( \overline{1} - t_\sigma (\overline{1} - \lambda) \) is the smallest \( \gamma \)-\(IF\) semiclosed set containing \( \lambda \) and so \( \overline{1} - t_\sigma (\overline{1} - \lambda) = c_\sigma (\lambda) \). Hence \( t'_\sigma = c_\sigma \).

(2) By Theorem 3.35 (7) and (1), \((c_\sigma)' = (t_\sigma)'' = t_\sigma\). This proves (2).

(3) If \( \lambda \in F \), then \((t_\sigma)' (\lambda) = \overline{1} - t_\sigma (\overline{1} - \lambda)\) and so by (2), \( c_\sigma (\lambda) = \overline{1} - t_\sigma (\overline{1} - \lambda)\) which implies that \( t_\sigma (\overline{1} - \lambda) = \overline{1} - c_\sigma (\lambda) \) for every \( \lambda \in F \).

(4) The proof is similar to the proof of (3).

The following Theorem 4.5 and Theorem 4.6 (1) give characterizations of \( \gamma \)-\(fuzzy\) semiopen sets in terms of \( \gamma \)-\(IF\) interior and \( \gamma \)-\(IF\) closure operators. Theorem 4.6 (2), (3) and (4) give properties of \( \gamma \)-\(IF\) semiinterior and \( \gamma \)-\(IF\) semiclosure operators.

Theorem 4.5. Let \( X \) be a nonempty set, \( \gamma \in \Gamma \) and \( \lambda \in F \). The following are equivalent:

1. \( \lambda \) is \( \gamma \)-\(IF\) semiopen.
2. \( \lambda \leq c_\gamma t_\gamma (\lambda) \).
3. \( c_\gamma (\lambda) = c_\gamma t_\gamma (\lambda) \).

Proof. (1) \( \Rightarrow \) (2) Suppose \( \lambda \) is \( \gamma \)-\(IF\) semiopen. Then there exists a \( \gamma \)-\(IF\) open set \( \mu \) such that \( \mu \leq \lambda \leq c_\gamma (\mu) \). Since \( \mu \) is \( \gamma \)-\(IF\) open, \( \mu = t_\gamma (\mu) \) and so \( \lambda \leq c_\gamma t_\gamma (\mu) \). Since \( c_\gamma t_\gamma \in \Gamma \), by Theorem 3.38 (1), and \( \mu \leq \lambda \), it follows that \( \lambda \leq c_\gamma t_\gamma (\lambda) \), which proves (2).

(2) \( \Rightarrow \) (3) Since \( c_\gamma \in \Gamma \) and \( t_\gamma (\lambda) \leq \lambda \), we have \( c_\gamma t_\gamma (\lambda) \leq c_\gamma (\lambda) \). By hypothesis and Theorem 3.38 (7), \( c_\gamma (\lambda) \leq c_\gamma c_\gamma t_\gamma (\lambda) = c_\gamma t_\gamma (\lambda) \). Therefore, \( c_\gamma (\lambda) = c_\gamma t_\gamma (\lambda) \).

(3) \( \Rightarrow \) (1) Since \( t_\gamma (\lambda) \) is a \( \gamma \)-\(IF\) open set such that \( t_\gamma (\lambda) \leq \lambda \leq c_\gamma t_\gamma (\lambda) \), \( \lambda \) is \( \gamma \)-\(IF\) semiopen.

Theorem 4.6. Let \( X \) be a nonempty set, \( \gamma \in \Gamma \) and \( \lambda \in F \). Then the following hold:

1. \( \lambda \) is \( \gamma \)-\(IF\) semiopen if and only if \( \lambda \) is \( c_\gamma t_\gamma \)-\(IF\) open if and only if \( \lambda = t_\gamma c_\gamma (\lambda) \).
2. \( t_\sigma = t_\gamma \) and \( c_\sigma = c_\gamma \).
3. \( t_\sigma (\lambda) = \lambda \wedge c_\gamma t_\gamma (\lambda) \).
(4) \( c_\gamma (\lambda) = \lambda \lor t_\gamma c_\gamma (\lambda) \).

**Proof.** The proof of (1) follows from Theorem 4.5 (1) and (2). (2) If \( x \in t_\sigma (\lambda) \), then there exists a \( \gamma \text{-IF} \) semiopen set \( \mu \) such that \( x \in \mu \leq \lambda \). By (1), \( \mu \) is a \( c_\gamma t_\gamma \text{-IF} \) open set and so \( x \in t_\gamma c_\gamma (\mu) \). Hence \( t_\sigma (\lambda) \leq t_\gamma c_\gamma (\lambda) \). Similarly, we can prove that \( t_\gamma c_\gamma (\lambda) \leq t_\sigma (\lambda) \). Therefore, \( t_\gamma = t_\sigma \). Again, \( c_\sigma = (t_\sigma)^\gamma \), by Theorem 4.4 (1) and so \( c_\sigma = (t_\gamma c_\gamma)^\gamma \) = \( c_\gamma \), by Theorem 3.35 (7). (3) Since \( t_\sigma (t_\sigma \lambda) = t_\sigma \lambda \) and \( t_\sigma \lambda \subseteq t_\gamma c_\gamma \lambda \) for every intuitionistic fuzzy subset \( \lambda \in \mathcal{F} \). Then we have \( t_\gamma c_\gamma (\lambda) = \lambda \land c_\gamma t_\gamma \lambda \) and so by, (2), \( t_\gamma \lambda = \lambda \land c_\gamma t_\gamma \lambda \). (4) Since \( t_\gamma (\lambda) = \lambda \land c_\gamma t_\gamma (\lambda) \), \( c_\gamma (\lambda) = \lambda \lor (c_\gamma t_\gamma) (\lambda) = \lambda \lor (c_\gamma)^\gamma (t_\gamma) (\lambda) = \lambda \lor t_\gamma c_\gamma (\lambda) \), by Theorem 3.38 (3) and Theorem 3.35 (7). By (2), \( c_\sigma (\lambda) = \lambda \lor t_\gamma c_\gamma (\lambda) \).

**Theorem 4.7.** If \( X \) is nonempty set, \( \gamma \in \Gamma, \lambda \in \mathcal{F}, \lambda \leq \mu \leq c_\gamma (\lambda) \) and \( \lambda \) is \( \gamma \text{-IF} \) semiopen, then \( \mu \) is \( \gamma \text{-IF} \) semiopen. In particular, the \( \gamma \) -closure of every \( \gamma \text{-IF} \) semiopen set is a \( \gamma \text{-IF} \) semiopen set.

**Proof.** Since \( \lambda \) is \( \gamma \text{-IF} \) semiopen, by Theorem 4.5 (3), \( c_\gamma (\lambda) = c_\gamma t_\gamma (\lambda) \) and so \( c_\gamma (\lambda) \leq c_\gamma t_\gamma (\mu) \). Since \( \mu \leq c_\gamma (\lambda) \), \( \mu \leq c_\gamma t_\gamma (\mu) \) and so by Theorem 4.5, \( \mu \) is \( \gamma \text{-IF} \) semiopen.

The following Theorem 4.8 gives characterizations of \( \gamma \text{-IF} \) semiclosed sets.

**Theorem 4.8.** Let \( X \) be a nonempty set, \( \gamma \in \Gamma \) and \( \gamma \in \mathcal{F} \). Then the following are equivalent.

1. \( \lambda \) is \( \gamma \text{-IF} \) semiclosed.
2. \( t_\gamma c_\gamma (\lambda) \leq \lambda \).
3. \( t_\gamma c_\gamma (\lambda) = t_\gamma (\lambda) \).
4. There exists a \( \gamma \text{-IF} \) closed set \( v \) such that \( t_\gamma (v) \leq \lambda \leq v \).

**Proof.** (1) \( \Rightarrow \) (2) If \( \lambda \) is \( \gamma \text{-IF} \) semiclosed, then \( \overline{\lambda} - \lambda \) is \( \gamma \text{-IF} \) semiopen and so \( \overline{\lambda} - \lambda \leq c_\gamma t_\gamma (\overline{\lambda} - \lambda) \), by Theorem 4.5 (2). By Theorem 3.35 (9) and (10), it follows that \( c_\gamma t_\gamma (\overline{\lambda} - \lambda) = \overline{\lambda} - \lambda c_\gamma (\lambda) \) and so \( t_\gamma c_\gamma (\lambda) \leq \lambda \). (2) \( \Rightarrow \) (3) \( t_\gamma c_\gamma (\lambda) \leq \lambda \) implies that \( t_\gamma c_\gamma (\lambda) \leq \lambda \). (3) \( \Rightarrow \) (4). If \( v = c_\gamma (\lambda) \), then \( v \) is a \( \gamma \text{-IF} \) closed set such that \( t_\gamma (v) = t_\gamma c_\gamma (\lambda) \) = \( t_\gamma (\lambda) \leq \lambda \leq v \), which prove (4). (4) \( \Rightarrow \) (1). If there exists a \( \gamma \text{-IF} \) closed set \( v \) such that \( t_\gamma (v) \leq \lambda \leq v \), then \( \overline{\lambda} - v \leq \overline{\lambda} - \lambda \leq \overline{\lambda} - v \) = \( c_\gamma (\overline{\lambda} - v) \). Since \( \overline{\lambda} - v \) is \( \gamma \text{-IF} \) open, \( \overline{\lambda} - \lambda \) is \( \gamma \text{-IF} \) semiopen and so \( \lambda \) is \( \gamma \text{-IF} \) semiclosed.

**References**
More on Generalized Intuitionistic Fuzzy Topology

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Abstract:
The purpose of the paper is to investigate some results concerning particular monotonic functions in generalized intuitionistic fuzzy topology.

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1. Introduction

After the introduction of fuzzy sets by Zadeh [6], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [1] is one among them. Using the notion of intuitionistic fuzzy sets, Coker [3] introduced the notion of intuitionistic fuzzy topological spaces. The aim of this paper is to present a common approach allowing to obtain families of intuitionistic fuzzy sets in an intuitionistic fuzzy topological space.

2. Preliminaries

Definition 2.1 [1]. Let $A$ be a nonempty fixed set. An intuitionistic fuzzy set ($IFS$, for short) $A$ is an object having the form $A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}$ where the functions $\mu_A : X \rightarrow I$ and $\gamma_A : X \rightarrow I$ denote respectively the degree of membership
(namely $\mu_A(x)$) and the degree of non-membership (namely $\gamma_A(x)$) of each element $x \in X$ to the set $A$, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

Obviously, every fuzzy set $A$ on a nonempty set $X$ is an IFS having the form $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$.

**Definition 2.2** [1]. Let $X$ be a nonempty set and let the IFS’s $A$ and $B$ in the form $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$, $B = \{(x, \mu_B(x), \gamma_B(x)) : x \in X\}$. Let $\{A_j : j \in J\}$ be an arbitrary family of IFS’s in $(X, \tau)$. Then,

1. $A \leq B$ if and only if $\forall x \in X \ [\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \mu_B(x)]$;
2. $\overline{A} = \{(x, \gamma_A(x), \mu_A(x)) : x \in X\}$;
3. $\bigcap A_j = \{(x, \wedge \mu_A(x), \vee \gamma_A(x)) : x \in X\}$;
4. $\bigcup A_j = \{(x, \vee \mu_A(x), \wedge \gamma_A(x)) : x \in X\}$;
5. $1_X = \{(x, 1, 0) : x \in X\}$ and $0_X = \{(x, 0, 1) : x \in X\}$.

**Definition 2.3.** An intuitionistic fuzzy topology [3] (IFT, for short) on a nonempty set $X$ is a family $\tau$ of IFSs in $X$ satisfying the following axioms:

(i) $0_X, 1_X \in \tau$;
(ii) $A_1 \cap A_2 \in \tau$ for every $A_1, A_2 \in \tau$;
(iii) $\bigcup A_j \in \tau$ for any $\{A_j : j \in J\} \subseteq \tau$.

In this case, the ordered pair $(X, \tau)$ is called an intuitionistic fuzzy topological space (IFTS, for short) and each IFS in $\tau$ is known as an intuitionistic fuzzy open set (IFOS, for short) in $X$. The complement of an intuitionistic fuzzy open set is called an intuitionistic fuzzy closed set (IFCS, for short). The family of all IFOSs (resp. IFCSs) of $(X, \tau)$ is denoted by $IFO(X)$ (resp. $IFC(X)$).

**Definition 2.4** [3]. Let $(X, \tau)$ be an IFTS and let $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$ be an IFS in $X$. Then the intuitionistic fuzzy interior and intuitionistic fuzzy closure of $A$ is defined by $\text{Int}(A) = \bigcup \{G \in \text{IFOS} \text{ in } X \text{ and } G \subseteq A\}$ and $\text{Cl}(A) = \bigcap \{G \in \text{IFCS} \text{ in } X \text{ and } G \supseteq A\}$.

**Remark 2.5.** For any IFS $A$ in $(X, \tau)$, we have, $\text{Cl}(1 - A) = 1 - \text{Int}(A)$, $\text{Int}(1 - A) = 1 - \text{Cl}(A)$.

Let $X$ be a nonempty set and $\mathcal{F} = \{\lambda : X \rightarrow [0, 1]\}$ be the family of all intuitionistic fuzzy sets defined on $X$. Let $\gamma : \mathcal{F} \rightarrow \mathcal{F}$ be a function such that $\lambda \leq \mu$.
implies that $\gamma(\lambda) \leq \gamma(\mu)$ for every $\lambda, \mu \in \mathcal{F}$. That is, $\gamma$ is a monotonic function defined on $\mathcal{F}$ by $\Gamma(\mathcal{F})$ or simply $\Gamma$. We will define the following subclasses of $\Gamma$.

1. For every $\alpha \in [0,1]$, define $\Gamma_\alpha = \{ \gamma \in \Gamma | \gamma(x) = \alpha \}$ for every $x \in X$.

2. $\Gamma_\lambda = \{ \gamma \in \Gamma | \gamma^2(\lambda) = \gamma(\lambda) \}$ for every $\lambda \in \mathcal{F}$.

3. $\Gamma_\mu = \{ \gamma \in \Gamma | \lambda \leq \gamma(\lambda) \}$ for every $\lambda \in \mathcal{F}$.

4. $\Gamma_\nu = \{ \gamma \in \Gamma | \lambda \leq \nu(\lambda) \}$ for every $\lambda \in \mathcal{F}$.

5. $\Gamma_\xi = \{ \gamma \in \Gamma | \gamma(\lambda) \leq \nu(\lambda) \}$.

If $\Sigma$ is a collection of some of the symbols $\lambda$, $\mu$, $\nu$, and $\alpha \in [0,1]$, then $\Gamma_i = \{ \gamma \in \Gamma | \gamma \in \Gamma_i \}$ for every $i \in \Sigma$.

3. Main results

**Definition 3.1.** Let $t, \kappa \in \Gamma$ and $A$ be an intuitionistic fuzzy subset of $X$. Then two operations $t$ and $\kappa$ are said to be

1. $\alpha(t,\kappa)$-related for $A$ if $tA \leq \alpha \kappa A$;

2. $s(t,\kappa)$-related for $A$ if $tA \leq \kappa A$;

3. $\gamma(t,\kappa)$-related for $A$ if $tA \leq \kappa \gamma A$;

4. $\beta(t,\kappa)$-related for $A$ if $tA \leq \kappa \beta A$.

**Remark 3.2.** (1) Let $X$ be an intuitionistic fuzzy topological space, and let $t = \text{Int}$ and $\kappa = \text{Cl}$. Then for an intuitionistic fuzzy subset $A$ of $X$, if $A$ is $\alpha$-open (resp. preopen), then $t$ and $\kappa$ are $\alpha(t,\kappa)$-related (resp. $\gamma(t,\kappa)$-related) for $A$.

(2) Let $(X, \lambda)$ be a generalized intuitionistic topological space [4], and let $t = i_\lambda$ and $\kappa = c_\lambda$. Then for an intuitionistic fuzzy subset $A$, if $A$ is $\lambda$-open (resp. $\lambda$-preopen), then $t$ and $\kappa$ are $\alpha(t,\kappa)$-related (resp. $\gamma(t,\kappa)$-related) for $A$.

(3) In (2), since $t, \kappa \in \Gamma$, for the $t$-open set $A$ of $X$, if $A$ is $t \kappa$-open (resp. $\kappa$ $t$-open, $t \kappa$-open, and $\kappa \kappa$-open), then $t$ and $\kappa$ are $\alpha(t,\kappa)$-related (resp. $s(t,\kappa)$-related, $\gamma(t,\kappa)$-related, $\beta(t,\kappa)$-related) for $A$.

**Definition 3.3.** Let $t, \kappa \in \Gamma$ and $A$ be an intuitionistic fuzzy subset of $X$. Then two operations $t$ and $\kappa$ are said to be $\alpha(t,\kappa)$-related (resp. $s(t,\kappa)$-related, $\gamma(t,\kappa)$-related, $\beta(t,\kappa)$-related) if $\alpha(t,\kappa)$-related (resp. $s(t,\kappa)$-related, $\gamma(t,\kappa)$-related, $\beta(t,\kappa)$-related) for all intuitionistic fuzzy subsets $A$ in $X$.
Theorem 3.4. For \( t, \kappa \in \Gamma \), if \( t \) and \( \kappa \) are \( s(t, \kappa) \)-related and \( p(t, \kappa) \)-related, then they are \( \beta(t, \kappa) \)-related.

Proof. For an intuitionistic fuzzy subset \( A \) of \( X \), \( tA \leq t\kappa A \Rightarrow \kappa tA \leq \kappa t\kappa A \). From the \( s(t, \kappa) \)-relatedness, we have \( tA \leq \kappa tA \leq \kappa t\kappa A \). Hence \( t \) and \( \kappa \) are \( \beta(t, \kappa) \)-related.

Consider the following conditions: For \( t, \kappa \in \Gamma \), (R1) \( \kappa tA \leq \kappa A \) for every intuitionistic fuzzy subset \( A \) of \( X \). R(2) \( t\kappa A \leq tA \) for every intuitionistic fuzzy subset \( A \) of \( X \).

Theorem 3.5. Let \( t, \kappa \in \Gamma \). Then

1. if \( t \) and \( \kappa \) are \( \alpha(t, \kappa) \)-related and (R1), then they are \( p(t, \kappa) \)-related;
2. if \( t \) and \( \kappa \) are \( \alpha(t, \kappa) \)-related and (R2), then they are \( s(t, \kappa) \)-related.

Proof. (1) For an intuitionistic fuzzy subset \( A \) of \( X \), from (R1), \( tA \leq t\kappa tA \leq t\kappa A \). Hence \( p(t, \kappa) \)-relatedness is proved.

(2) For an intuitionistic fuzzy subset \( A \) of \( X \), from (R2), \( tA \leq t\kappa tA \leq t\kappa A \) and so they are \( s(t, \kappa) \)-related.

Corollary 3.6. Let \( t, \kappa \in \Gamma \). Then if \( t \) and \( \kappa \) are \( \alpha(t, \kappa) \)-related, (R1) and (R2), then they are \( \beta(t, \kappa) \)-related.

Proof. It follows from Theorem 3.4 and Theorem 3.5. Remark 3.7. Let \( t, \kappa \in \Gamma \).

1. \( s(t, \kappa) \)-relatedness \( \rightarrow \beta(t, \kappa) \)-relatedness.
2. \( \alpha(t, \kappa) \)-relatedness \( \overset{(R2)}{\Rightarrow} s(t, \kappa) \)-relatedness
3. \( \alpha(t, \kappa) \)-relatedness \( \overset{(R1)}{\Rightarrow} p(t, \kappa) \)-relatedness
4. \( \alpha(t, \kappa) \)-related + (R1) + (R2) \( \Rightarrow \beta(t, \kappa) \)-relatedness.

Theorem 3.8. Let \( \kappa \in \Gamma^+ \). Then for every \( t \in \Gamma \), \( t \) and \( \kappa \) are

1. \( p(t, \kappa) \)-related;
2. \( s(t, \kappa) \)-related;
3. \( \beta(t, \kappa) \)-related;
4. if \( s(t, \kappa) \)-related, then \( \beta(t, \kappa) \)-related;
5. if \( p(t, \kappa) \)-related, then \( \beta(t, \kappa) \)-related.

Proof. (1) For every intuitionistic fuzzy subset \( A \) of \( X \), easily we have \( A \leq \kappa A \Rightarrow tA \leq t\kappa A \).

(2) For every intuitionistic fuzzy subset \( A \) of \( X \), \( tA \leq \kappa tA \).

(3) For every intuitionistic fuzzy subset \( A \) of \( X \), \( tA \leq t\kappa A \leq \kappa t\kappa A \).
(4), (5) For every intuitionistic fuzzy subset \( A \) of \( X \), it follows \( tA \leq kA \leq t\kappa A \) and \( tA \leq t\kappa A \leq t\kappa A \).

**Remark 3.9.** For \( t \in \Gamma \), \( k \in \Gamma_+ \), from Remark 3.7 and Theorem 3.8,

\[
\begin{align*}
\alpha(t, \kappa) &\text{-relatedness } \overset{R2}{\longrightarrow} s(t, \kappa) \text{-relatedness } \overset{R1}{\longrightarrow} \beta(t, \kappa) \text{-relatedness.} \\
\alpha(t, \kappa) &\text{-relatedness } \overset{R2}{\longrightarrow} p(t, \kappa) \text{-relatedness } \overset{R1}{\longrightarrow} \beta(t, \kappa) \text{-relatedness.}
\end{align*}
\]

**Lemma 3.10.** Let \( t \in \Gamma_+ \). Then for every \( \kappa \in \Gamma \), (R1) and (R2) are fulfilled.

**Theorem 3.11.** Let \( t \in \Gamma_+ \). For every \( \kappa \in \Gamma \), if \( t \) and \( \kappa \) are \( \alpha(t, \kappa) \)-related, then they are

1. \( s(t, \kappa) \)-related;
2. \( p(t, \kappa) \)-related;
3. \( \beta(t, \kappa) \)-related.

**Remark 3.12.** Let \( t \in \Gamma_+ \), \( \kappa \in \Gamma \).

1. \( \alpha(t, \kappa) \)-relatedness \( \rightarrow \) \( s(t, \kappa) \)-relatedness \( \rightarrow \) \( \beta(t, \kappa) \)-relatedness.

2. If \( t \in \Gamma_+ \) and \( \kappa \in \Gamma_+ \), then by Theorem 3.8 and Theorem 3.11,

\[
\begin{align*}
\alpha(t, \kappa) \text{-relatedness } &\rightarrow s(t, \kappa) \text{-relatedness } \rightarrow \beta(t, \kappa) \text{-relatedness.} \\
\alpha(t, \kappa) &\text{-relatedness } \rightarrow p(t, \kappa) \text{-relatedness } \rightarrow \beta(t, \kappa) \text{-relatedness.}
\end{align*}
\]

**Theorem 3.13.** Let \( t \in \Gamma_+ \) and \( \kappa \in \Gamma \). If \( t \) and \( \kappa \) are \( s(t, \kappa) \)-related, then they are \( \alpha(t, \kappa) \)-related.

**Proof.** For every intuitionistic fuzzy subset \( A \) of \( X \), since \( uA = tA \), we have \( tA \leq kA \Rightarrow uA \leq t\kappa A \Rightarrow uA \leq t\kappa A \). Hence \( t \) and \( \kappa \) are \( \alpha(t, \kappa) \)-related.

**Theorem 3.14.** Let \( t \in \Gamma_+ \) and \( \kappa \in \Gamma \). If \( t \) and \( \kappa \) are \( p(t, \kappa) \)-related, then they are \( \alpha(t, \kappa) \)-related.

**Proof.** For a intuitionistic fuzzy subset \( A \) of \( X \), since \( tA \leq 1 \), by hypothesis, \( tA = t(tA) \leq t\kappa(tA) \). Therefore, \( t \) and \( \kappa \) are \( \alpha(t, \kappa) \)-related.

**Theorem 3.15.** Let \( t \in \Gamma_+ \) and \( \kappa \in \Gamma \). Then

1. If \( t \) and \( \kappa \) are \( s(t, \kappa) \)-related and (R1), then they are \( p(t, \kappa) \)-related;
(2) if \( t \) and \( \kappa \) are \( p(t,\kappa') \)-related and (R2), then they are \( s(t,\kappa') \)-related.

Proof. (1) For an intuitionistic fuzzy subset \( A \) of \( X \), \( tA \leq \kappa tA \Rightarrow tA = t\kappa A \leq t\kappa A \). Therefore, \( t \) and \( \kappa \) are \( p(t,\kappa') \)-related.

(2) For an intuitionistic fuzzy subset \( A \) of \( X \), since \( tA \leq 1 \), by \( p(t,\kappa') \)-relatedness, \( tA \leq \kappa tA \leq \kappa A \). From \( tA = \kappa A \), it follows that \( t \) and \( \kappa \) are \( s(t,\kappa') \)-related.

Lemma 3.16. Let \( t \in \Gamma_2 \) and \( \kappa \in \Gamma_1 \). Then \( t \) and \( \kappa \) are \( \alpha(t,\kappa') \)-related.

Proof. First, \( t \) and \( \kappa \) are \( s(t,\kappa') \)-related (or \( p(t,\kappa') \)-related) from Theorem 3.8. Thus from Theorem 3.13 (or Theorem 3.14), they are \( \alpha(t,\kappa') \)-related.

Remark 3.17. For \( t \in \Gamma_2 \), \( \kappa \in \Gamma \), we have the following:

1. \( s(t,\kappa') \)-relatedness \( \Rightarrow \alpha(t,\kappa') \)-relatedness
2. \( p(t,\kappa') \)-relatedness \( \Rightarrow \alpha(t,\kappa') \)-relatedness
3. If \( t \) and \( \kappa \) satisfy (R1), then: \( s(t,\kappa') \)-relatedness \( p(t,\kappa') \)-relatedness \( \alpha(t,\kappa') \)-relatedness.
4. If \( t \) and \( \kappa \) satisfy (R2), then: \( p(t,\kappa') \)-relatedness \( s(t,\kappa') \)-relatedness \( \alpha(t,\kappa') \)-relatedness.

Lemma 3.18. Let \( t \in \Gamma \) and \( \kappa \in \Gamma_2 \). If \( tA \leq \kappa A \), then

1. they satisfy (R1) and (R2);
2. \( \kappa t\kappa A \leq \kappa A \).

Proof. Obvious.

Lemma 3.19. Let \( t \in \Gamma \) and \( \kappa \in \Gamma_2 \). Then if \( t \) and \( \kappa \) are (R1) (or (R2)), for every intuitionistic fuzzy subset \( A \) of \( X \), then \( \kappa t\kappa A \leq \kappa A \).

Proof. Obvious.

Lemma 3.20. Let \( t \in \Gamma \) and \( \kappa \in \Gamma_2 \). If \( t \) and \( \kappa \) are (R1) (resp., (R2)) and \( s(t,\kappa') \)-related (resp., \( p(t,\kappa') \)-related), then \( tA \leq \kappa A \).

Lemma 3.21. Let \( t,\kappa \in \Gamma_2 \). If \( t \) and \( \kappa \) are (R1) and \( s(t,\kappa') \)-related, then

1. \( \kappa t\kappa A = t\kappa A \).
2. \( \kappa t\kappa A = \kappa tA \).

Theorem 3.22. Let \( t,\kappa \in \Gamma_2 \). If \( t \) and \( \kappa \) are \( \beta(t,\kappa') \)-related and (R1) (or (R2)), then they are

1. \( s(t,\kappa') \)-related;
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(2) $p(t, \kappa)$-related;

(3) $\alpha(t, \kappa)$-related.

Proof. (1) For an intuitionistic fuzzy subset $A$ of $X$, since $tA$ is an intuitionistic fuzzy subset of $X$, from the $\beta(t, \kappa)$-relatedness and the condition (R1), it follows $t(tA) \leq \kappa tA \leq \kappa tA$. Hence by $t \in \Gamma$, $t$ and $\kappa$ are $s(t, \kappa)$-related.

(2) For an intuitionistic fuzzy subset $A$ of $X$, from the $\beta(t, \kappa)$-relatedness and the condition (R1), it follows $tA \leq \kappa t \leq \kappa tA$. This implies $tA = tA \leq tA$. Hence $t$ and $\kappa$ are $p(t, \kappa)$-related.

(3) It follows from Theorem 3.13.

**Corollary 3.23.** Let $t, \kappa \in \Gamma$.

(1) If $t$ and $\kappa$ satisfy (R1), then:

<table>
<thead>
<tr>
<th>$\beta(t, \kappa)$-relatedness</th>
<th>$\alpha(t, \kappa)$-relatedness</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s(t, \kappa)$-relatedness</td>
<td>$p(t, \kappa)$-relatedness</td>
</tr>
</tbody>
</table>

(2) If $t$ and $\kappa$ satisfy (R2), then:

<table>
<thead>
<tr>
<th>$\beta(t, \kappa)$-relatedness</th>
<th>$\alpha(t, \kappa)$-relatedness</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(t, \kappa)$-relatedness</td>
<td>$s(t, \kappa)$-relatedness</td>
</tr>
</tbody>
</table>

Proof. (1) It follows from Theorem 3.4, Theorem 3.15 (1), Remark 3.17 and Theorem 3.22.

(2) It follows from Theorem 3.4, Theorem 3.15 (2), Remark 3.17 and Theorem 3.22.

**Remark 3.24.** Let $t, \kappa \in \Gamma$. If $t \in \Gamma_-$, then by Lemma 3.10, $\alpha(t, \kappa)$-related $\iff s(t, \kappa)$-related $\iff p(t, \kappa)$-related $\iff \beta(t, \kappa)$-related.

4. $\alpha(t, \kappa)$-open, $s(t, \kappa)$-open, $p(t, \kappa)$-open, $\beta(t, \kappa)$-open

**Definition 4.1.** Let $t, \kappa \in \Gamma$ and $A$ be an intuitionistic fuzzy subset of $X$. Then $A$ is said to be (1) $\alpha(t, \kappa)$-open if $tA \leq \kappa tA$; (2) $s(t, \kappa)$-open if $tA \leq \kappa tA$; (3) $p(t, \kappa)$-open if $tA \leq \kappa tA$; (4) $\beta(t, \kappa)$-open if $tA \leq \kappa tA$.

**Remark 4.2** (1) Let $(X, s)$ be an intuitionistic fuzzy topological space, and let $t = \text{Int}$ and $\kappa = \text{Cl}$. Then for an intuitionistic fuzzy subset $A$ of $X$, if $A$ is $\alpha$-open (resp., preopen), then it is $\alpha(t, \kappa)$-open (resp. $p(t, \kappa)$-open).
(2) Let \((X, \lambda)\) be a generalized topological space, and let \(t = i_\lambda\) and \(\kappa = c_\lambda\). Then for an intuitionistic fuzzy subset \(A\) of \(X\), if \(A\) is \(t\)-\(\alpha\)-open (resp. \(t\)-preopen), then it is \(\alpha(t, \kappa)\)-open (resp. \(p(t, \kappa)\)-open). Furthermore, if \(A\) is a \(\lambda\)-open set, then \(A\) is also \(\alpha(t, \kappa)\)-open \(s(t, \kappa)\)-open, \(p(t, \kappa)\)-open, \(\beta(t, \kappa)\)-open).

**Lemma 4.3.** Let \(t, \kappa \in \Gamma\). Then every \(\kappa\)-open set is \(\alpha(t, \kappa)\)-open.

**Theorem 4.4.** Let \(t, \kappa \in \Gamma\) and \(A\) be an intuitionistic fuzzy subset of \(X\). If \(t \in \Gamma_{-}\), then the following hold:

1. \(t\kappa\)-open \(\Rightarrow \alpha(t, \kappa)\)-open,
2. \(t\kappa\)-open \(\Rightarrow p(t, \kappa)\)-open.

**Theorem 4.5.** Let \(t, \kappa \in \Gamma\) and \(A\) be an intuitionistic fuzzy subset of \(X\). If \(t, \kappa\) satisfy (R2), then the following hold:

1. \(t\kappa\)-open \(\Rightarrow \alpha(t, \kappa)\)-open,
2. \(\kappa\kappa\)-open \(\Rightarrow s(t, \kappa)\)-open,
3. \(t\kappa\)-open \(\Rightarrow p(t, \kappa)\)-open,
4. \(\kappa\kappa\)-open \(\Rightarrow \beta(t, \kappa)\)-open.

**Corollary 4.6.** Let \(t, \kappa \in \Gamma\) and \(A\) be an intuitionistic fuzzy subset of \(X\). If \(t \in \Gamma_{-}\), then the following hold:

1. \(t\kappa\)-open \(\Rightarrow \alpha(t, \kappa)\)-open;
2. \(\kappa\kappa\)-open \(\Rightarrow s(t, \kappa)\)-open;
3. \(t\kappa\)-open \(\Rightarrow p(t, \kappa)\)-open;
4. \(\kappa\kappa\)-open \(\Rightarrow \beta(t, \kappa)\)-open.

**Lemma 4.7.** Let \(t, \kappa \in \Gamma\). For an \(t\)-open set \(A\) of \(X\), if \(A\) is \(\alpha(t, \kappa)\)-open (resp. \(s(t, \kappa)\)-open, \(p(t, \kappa)\)-open, \(\beta(t, \kappa)\)-open), then it is \(t\kappa\)-open (resp. \(\kappa\kappa\)-open, \(t\kappa\)-open, \(\kappa\kappa\)-open).

**Remark 4.8.**

1. For \(t, \kappa \in \Gamma\), \(\left\{ \begin{array}{l} s(t, \kappa)\text{-open} \\ p(t, \kappa)\text{-open} \end{array} \right\} \Rightarrow \beta(t, \kappa)\)-open. In [4], the authors have shown that if \(t \in \Gamma_{-}\) and \(k \in \Gamma_{-}\), then: \(t\kappa\rightarrow \left\{ \begin{array}{l} \kappa, \kappa\text{-open} \\ t, \kappa\text{-open} \end{array} \right\} \Rightarrow \kappa\kappa\)-open.

2. If \(t \in \Gamma_{-}\) and \(\kappa \in \Gamma\), then: \(\alpha(t, \kappa)\)-open \(\left\{ \begin{array}{l} s(t, \kappa)\text{-open} \\ p(t, \kappa)\text{-open} \end{array} \right\} \Rightarrow \beta(t, \kappa)\)-open.
(4) If \( t \in \Gamma_- \) and \( \kappa \in \Gamma_+ \), then: \( t \)-open \( \rightarrow \alpha(t,\kappa) \rightarrow s(t,\kappa) \)-open \( \rightarrow \beta(t,\kappa) \)-open.

Furthermore, since a fuzzy subset \( A \) is \( t \)-open if and only if \( tA = A \), we have \( t \)-open \( \rightarrow t\alpha \)-open \( \rightarrow t\kappa \)-open \( \rightarrow t\kappa \)-open.

References

Characterization of Connectedness and Surroundness in An Intuitionistic Fuzzy Digital Topology

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Abstract:
Topological relationships among parts of a digital picture, such as connectedness and surroundedness, play an important role in picture analysis and description. This paper generalizes these concepts to an intuitionistic fuzzy sets and discussed some of their basic properties.

Keywords:
Intuitionistic fuzzy strength of a path, intuitionistic fuzzy connectedness, intuitionistic fuzzy components like intuitionistic fuzzy plateau, intuitionistic fuzzy top, intuitionistic fuzzy bottom and intuitionistic fuzzy surroundness.

1. Introduction
The concept of fuzzy sets was introduced by Zadeh [5] and later Atanassov [1] generalized the idea to intuitionistic fuzzy sets. Geometrical properties and relationships among parts of a digital picture play an important role in picture analysis and description that was studied by A. Rosenfeld and A. Kak [2,3]. Later A. Rosenfeld [4] extend the concepts of digital picture geometry to fuzzy sets. In this paper, the topological concepts of connectedness and surroundedness with respect to an intuitionistic fuzzy sets are introduced. Some interesting properties are established.

2. Preliminaries
Let $\Sigma$ be a rectangular array of integer-coordinate points. Thus the point $P = (x, y)$ of $\Sigma$ has four horizontal and vertical neighbors, namely $(x \pm 1, y)$ and $(x, y \pm 1)$; and it also has four diagonal neighbors, namely $(x \pm 1, y \pm 1)$ and $(x \pm 1, y \mp 1)$. We say that former points are 4-adjacent to, or 4-neighbors of $P$ and we say that both types of neighbors are 8-adjacent to, or 8-neighbors of $P$. Note that if $P$ is on the border $B$ of $\Sigma$, some of these neighbors may not exist.
For all points \( P, Q \) of \( \Sigma \), by a path \( \rho \) from \( P \) to \( Q \) we mean a sequence of points
\[
P = P_0, P_1, \ldots, P_n = Q
\]
such that \( P_i \) is adjacent to \( P_{i+1} \), \( 1 \leq i \leq n \). Note that this is two definitions in one (“4-path” and “8-path”), depending on whether “adjacent” means “4-adjacent” or “8-adjacent”. The same is true for many of the definitions that follow, but we usually do not mention this. Let \( S \) be any subset of \( \Sigma \). We say that the points \( P, Q \) of \( \Sigma \) are connected in \( S \) if there is a path from \( P \) to \( Q \) consisting entirely of points of \( S \). Readily, “connected” is an equivalence relation: \( P \) is connected to \( P \) (by a path of length 0); if \( P \) is connected to \( Q \) then \( Q \) is connected to \( P \) (the reversal of a path is a path) and if \( P \) is connected to \( Q \) and \( Q \) to \( R \), then \( P \) is connected to \( R \) (the concatenation of two paths is a path). This relation partitions \( S \) into equivalence classes, which are maximal subsets \( S' \) of \( S \) such that every \( P, Q \) belonging to a given \( S' \) are connected. These classes are called the (connected) components of \( S \).

Let \( \overline{S} = \Sigma - S \) be the complement of \( S \). We assume, for simplicity, that the border points of \( \Sigma \) are all in \( \overline{S} \). Thus one component of \( \overline{S} \) always contains the border \( B \) of \( \Sigma \). The other components, if any, are called holes in \( S \). (More generally, for any subsets \( U, V, W \) of \( \Sigma \), we say that \( V \) separates \( U \) from \( W \) if any path from \( U \) to \( W \) must meet \( V \). Thus \( S \) surrounds \( T \) if it separates \( T \) from the border \( B \) of \( \Sigma \).)

**Definition 2.1** \[1\]. Let \( X \) be a nonempty fixed set and \( I \) be the closed interval \([0,1]\). An intuitionistic fuzzy set (IFS) \( A \) is an object of the following form
\[
A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\},
\]
where the mappings \( \mu_A : X \to I \) and \( \gamma_A : X \to I \) denote the degree of membership (namely \( \mu_A(x) \)) and the degree of nonmembership (namely \( \gamma_A(x) \)) for each element \( x \in X \) to the set \( A \), respectively, and \( 0 \leq \mu_A(x) + \gamma_A(x) \leq 1 \) for each \( x \in X \). Obviously, every fuzzy set \( A \) on a nonempty set \( X \) is an IFS of the following form, \( A = \{(x, \mu_A(x), 1 - \mu_A(x)) : x \in X\} \). For the sake of simplicity, we shall use the symbol \( A = (x, \mu_A(x), \gamma_A(x)) \) for the intuitionistic fuzzy set \( A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\} \).

**Definition 2.2** \[1\]. Let \( X \) be a nonempty set and the IFSs \( A \) and \( B \) in the form
\[
A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}, \quad B = \{(x, \mu_B(x), \gamma_B(x)) : x \in X\}.
\]
Then
(i) \( A \subseteq B \) iff \( \mu_A(x) \leq \mu_B(x) \) and \( \gamma_A(x) \geq \gamma_B(x) \) for all \( x \in X \);
(ii) \( \overline{A} = \{(x, \gamma_A(x), \mu_A(x)) : x \in X\} \);
(iii) \( A \cap B = \{(x, \mu_A(x) \land \mu_B(x), \gamma_A(x) \lor \gamma_B(x)) : x \in X\} \).
(iv) \( A \cup B = \left\{ \left( x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x) \right) : x \in X \right\} \).

**Definition 2.3** [1]. The IFSs 0 and 1 are defined by 0 = \( \left\{ \langle x, 0, 1 \rangle : x \in X \right\} \) and 1 = \( \left\{ \langle x, 1, 0 \rangle : x \in X \right\} \).

3. Properties of an intuitionistic fuzzy connectedness in an intuitionistic fuzzy digital topology

**Definition 3.1.** Let \( \Sigma \) be a rectangular array of integer-coordinate points. Let \( A = \left\{ P, \mu_A, \gamma_A \right\} \) be an intuitionistic fuzzy set in \( \Sigma \). Let \( \rho : P = P_0, P_1, \ldots, P_n = Q \) be any path between two points of \( \Sigma \). Then an intuitionistic fuzzy strength of a path with respect to an intuitionistic fuzzy set \( A \) in \( \Sigma \) is defined and denoted by \( s_A \) with

\[
\mu_{s_A}(\rho) = \min_{0 \leq i \leq n} \mu_A(P_i) \quad \text{and} \quad \gamma_{s_A}(\rho) = \max_{0 \leq i \leq n} \gamma_A(P_i)
\]

such that \( \mu_{s_A}(\rho) + \gamma_{s_A}(\rho) \leq 1 \).

**Definition 3.2.** Let \( \Sigma \) be a rectangular array of integer-coordinate points. Let \( A = \left\{ P, \mu_A, \gamma_A \right\} \) be an intuitionistic fuzzy set in \( \Sigma \). Let \( \rho : P = P_0, P_1, \ldots, P_n = Q \) be any path between two points of \( \Sigma \). Then an intuitionistic fuzzy connectedness of the points \( P \) and \( Q \) of a path with respect to an intuitionistic fuzzy set \( A \) in \( \Sigma \) is defined and denoted by \( C_A \) with

\[
\mu_{C_A}(P, Q) = \min_{\rho} \mu_A(P) \quad \text{and} \quad \gamma_{C_A}(P, Q) = \min_{\rho} \gamma_A(P)
\]

such that \( \mu_{C_A}(P, Q) + \gamma_{C_A}(P, Q) \leq 1 \). Clearly, \( C_A \) is an intuitionistic fuzzy set in \( \Sigma \times \Sigma \).

**Proposition 3.1.** Let \( \Sigma \) be a rectangular array of integer-coordinate points. Let \( A = \left\{ P, \mu_A, \gamma_A \right\} \) be an intuitionistic fuzzy set in \( \Sigma \). Then for all \( P, Q \) in \( \Sigma \),

(i) \( \mu_{C_i}(P, Q) = \mu_A(P) \) and \( \gamma_{C_i}(P, Q) = \gamma_A(P) \).

(ii) \( \mu_{C_i}(P, Q) = \mu_{C_i}(Q, P) \) and \( \gamma_{C_i}(P, Q) = \gamma_{C_i}(Q, P) \).

**Proof.** (i) For any path \( \rho : P = P_0, P_1, \ldots, P_n = Q \), the degree of membership of an intuitionistic fuzzy strength of a path with respect to an intuitionistic fuzzy set \( A \) in \( \Sigma \) is always less than or equal to the degree of membership of an intuitionistic fuzzy set \( A \). That is, \( \mu_{s_A}(\rho) \leq \mu_A(P) \) and the degree of nonmembership of an intuitionistic fuzzy strength of a path with respect to an intuitionistic fuzzy set \( A \) in \( \Sigma \) is always greater than or equal to the degree of nonmembership of an intuitionistic fuzzy set \( A \). That is, \( \gamma_{s_A}(\rho) \geq \gamma_A(P) \). Now, let \( \rho \) be any path from \( P \) to \( P \) itself. Thus \( \mu_{C_i} = \mu_A(P) \) and \( \gamma_{C_i} = \gamma_A(P) \), for \( P \in \Sigma \) and \( i = 1, 2, 3, \ldots, n \). This implies that \( \mu_A(P) = \max_{\rho} \mu_{s_A}(\rho) = \mu_{C_i}(P, P) \) and \( \gamma_A(P) = \min_{\rho} \gamma_{s_A}(\rho) = \gamma_{C_i}(P, P) \).

(ii) it follows from the fact that the reversal of a path is a path and reverse path preserves path strength.
Remark 3.1. Let \( \Sigma \) be a rectangular array of integer-coordinate points. Let \( A = \{P, \mu_A, \gamma_A\} \) be an intuitionistic fuzzy set in \( \Sigma \). Let \( \Omega = \{P \in \Sigma | \mu_A(P) = 1 \text{ and } \gamma_A(P) = 0\} \). Let \( \rho: P = P_0, P_1, \cdots, P_n = Q \) be any path between two points of \( \Sigma \). Let \( s_A \) and \( C_A \) be an intuitionistic fuzzy strength of a path and intuitionistic fuzzy connectedness of the points \( P \) and \( Q \) of a path with respect to an intuitionistic fuzzy set \( A \) in \( \Sigma \). Then

(i) \( \mu_{s_A}(\rho) = 1 \) and \( \gamma_{s_A}(\rho) = 0 \) if and only if a path \( \rho \) contains entirely of points of \( \Omega \).

(ii) \( \mu_{C_A}(P,Q) = 1 \) and \( \gamma_{C_A}(P,Q) = 0 \) if and only if \( P \) and \( Q \) are intuitionistic fuzzy connected in \( \Omega \).

Proposition 3.2. Let \( \Sigma \) be a rectangular array of integer-coordinate points. Let \( A = \{P, \mu_A, \gamma_A\} \) be an intuitionistic fuzzy set in \( \Sigma \). Then for all \( P, Q \in \Sigma \), we have

\[
\mu_{C_A}(P,Q) \leq \min(\mu_A(P), \mu_A(Q)) \quad \text{and} \quad \gamma_{C_A}(P,Q) \geq \max(\gamma_A(P), \gamma_A(Q)).
\]

Proof. Let \( \rho: P = P_0, P_1, \cdots, P_n = Q \) be any path. Then the degree of membership of an intuitionistic fuzzy strength of a path \( \rho \) with respect to an intuitionistic fuzzy set \( A \) in \( \Sigma \) is given by

\[
\mu_{C_A}(\rho) = \mu_{\min} \quad \text{and} \quad \gamma_{C_A}(\rho) = \mu_{\max},
\]

and the degree of nonmembership of an intuitionistic fuzzy strength of a path \( \rho \) with respect to an intuitionistic fuzzy set \( A \) in \( \Sigma \) is given by

\[
\mu_{C_A}(\rho) = \mu_{\max} \quad \text{and} \quad \gamma_{C_A}(\rho) = \mu_{\min}.
\]

This implies that \( C_A \) is an intuitionistic fuzzy connected of \( P \) and \( Q \) with respect to an intuitionistic fuzzy set \( A \) whose \( \gamma_{C_A}(P,Q) = \mu_{\min} \) and \( \gamma_{C_A}(P,Q) = \mu_{\min} \).

Definition 3.3. Let \( \Sigma \) be a rectangular array of integer-coordinate points. Let \( A = \{P, \mu_A, \gamma_A\} \) be an intuitionistic fuzzy set in \( \Sigma \). Let \( \Omega = \{P \in \Sigma | \mu_A(P) = 1 \text{ and } \gamma_A(P) = 0\} \). Let \( \rho: P = P_0, P_1, \cdots, P_n = Q \) be any path between two points of \( \Sigma \). Let \( \Gamma \subseteq \Sigma \) be any subset of \( \Sigma \). Then intuitionistic fuzzy connectedness of \( \Gamma \) with respect to an intuitionistic fuzzy set \( A \) in \( \Sigma \) is defined and denote by \( \mu_{C_A}(\Gamma, \rho) = \mu_{\min} \) and \( \gamma_{C_A}(\Gamma, \rho) = \mu_{\max} \).

Corollary 3.1. Let \( \Sigma \) be a rectangular array of integer-coordinate points. Let \( A = \{P, \mu_A, \gamma_A\} \) be an intuitionistic fuzzy set in \( \Sigma \). Let \( \Gamma \subseteq \Sigma \) be any subset of \( \Sigma \). Then \( \mu_{C_A}(\Gamma, \rho) = \mu_{\min} \) and \( \gamma_{C_A}(\Gamma, \rho) = \mu_{\max} \).
Proof. The proof is obvious.

Definition 3.4. Let \( \Sigma \) be a rectangular array of integer-coordinate points. Let \( \rho: P = P_0, P_1, \ldots, P_n = Q \) be any path between two points of \( \Sigma \). Let \( A \) be an intuitionistic fuzzy set in \( \Sigma \). Then \( P \) and \( Q \) are intuitionistic fuzzy connected with respect to an intuitionistic fuzzy set \( A \) if

\[
\mu_{c_i}(P, Q) = \min \{\mu_A(P), \mu_A(Q)\} \quad \text{and} \quad \gamma_{c_i}(P, Q) = \max \{\gamma_A(P), \gamma_A(Q)\}
\]

Proposition 3.3. Let \( \Sigma \) be a rectangular array of integer-coordinate points. Let \( A = \{P, \mu_A, \gamma_A\} \) be an intuitionistic fuzzy set in \( \Sigma \). Let \( \rho: P = P_0, P_1, \ldots, P_n = Q \) be any path between two points of \( \Sigma \). Then \( P \) and \( Q \) are intuitionistic fuzzy connected in an intuitionistic fuzzy set \( A \) if and only if there exists a path \( \rho': P = P_0, P_1, \ldots, P_n = Q \) such that \( \mu_A(P) \geq \min \{\mu_A(P), \mu_A(Q)\} \) and \( \gamma_A(P) \leq \max \{\gamma_A(P), \gamma_A(Q)\} \) for \( 0 \leq i \leq n \).

Proof. Suppose that if exists such a path \( \rho': P = P_0, P_1, \ldots, P_n = Q \), we have \( \mu_{c_i}(P, Q) = \max \mu_A(\rho) \geq \mu_A(P) \approx \min \{\mu_A(P), \mu_A(Q)\} \) and \( \gamma_{c_i}(P, Q) = \min \gamma_A(\rho') \leq \max \gamma_A(P), \gamma_A(Q) \). By Proposition 3.2, \( \mu_{c_i}(P, Q) = \min \{\mu_A(P), \mu_A(Q)\} \) and \( \gamma_{c_i}(P, Q) = \max \{\gamma_A(P), \gamma_A(Q)\} \). This implies that \( P \) and \( Q \) are intuitionistic fuzzy connected in an intuitionistic fuzzy set \( A \).

Conversely, if \( P \) and \( Q \) are intuitionistic fuzzy connected in an intuitionistic fuzzy set \( A \). Let \( \rho': P = P_0, P_1, \ldots, P_n = Q \) be a path, for which \( \mu_A(\rho') = \max \mu_A(\rho) = \mu_A(P, Q) = \min \mu_A(P), \mu_A(Q) \) and \( \gamma_A(\rho') = \min \gamma_A(\rho) = \gamma_A(P, Q) = \max \gamma_A(P), \gamma_A(Q) \). Then \( \mu_A(P_i) \geq \min \mu_A(P) = \mu_A(\rho') \) and \( \gamma_A(P_i) \leq \max \gamma_A(P) = \gamma_A(\rho') \) for all \( P_i \) on a path \( \rho' \). Hence the proof is completed.

Clearly, if intuitionistic fuzzy set \( A \) in \( \Sigma \) with \( \mu_A(P) = 1 = \mu_A(Q) \) and \( \gamma_A(P) = 0 = \gamma_A(Q) \), then \( P \) and \( Q \) are intuitionistic fuzzy connected if and only if there exists a path \( \rho \) from \( P \) to \( Q \) such that for any point \( P' \) in \( \rho \), \( \mu_A(P') = 1 \) and \( \gamma_A(P') = 0 \).

Proposition 3.4. Let \( \Sigma \) be a rectangular array of integer-coordinate points. Let \( \rho: P = P_0, P_1, \ldots, P_n = Q \) be any path between two points of \( \Sigma \). Let \( A \) be an intuitionistic fuzzy set in \( \Sigma \). Let \( \mathcal{C}^A = \{(P, Q) \mid P \) and \( Q \) are intuitionistic fuzzy connected in \( A \} \). Then \( \mathcal{C}^A \) is reflexive and symmetric but not necessarily transitive with respect to an intuitionistic fuzzy set \( A \).
Proof.  Reflexive. For all \( P \in \Sigma \), \( \mu_{\Sigma_i}(P,P) = \mu_A(P) = \min(\mu_A(P), \mu_A(P)) \) and \( \gamma_{\Sigma_i}(P,P) = \gamma_A(P) = \max(\gamma_A(P), \gamma_A(P)) \). This implies that \( \mathcal{C}^A \) is reflexive with respect to an intuitionistic fuzzy set \( A \).

Symmetric. For all \( P, Q \in \Sigma \), \( \mu_{\Sigma_i}(P,Q) = \mu_A(Q) \) and \( \mu_A(Q) = \mu_{\Sigma_i}(Q,P) \) and \( \gamma_{\Sigma_i}(P,Q) = \gamma_A(Q) \) and \( \gamma_A(Q) = \gamma_{\Sigma_i}(Q,P) \). This implies that \( \mathcal{C}^A \) is symmetric with respect to an intuitionistic fuzzy set \( A \).

Not Transitive. Let \( \Sigma \) be the 1-by-3 array \( P, Q, R \). Let \( (P,Q) \) and \( (Q,R) \) be intuitionistic fuzzy connected in \( A \) respectively and let \( \mu_A(P) = 1 = \mu_A(R), \mu_A(Q) < 1 \) and \( \gamma_A(P) = 0 = \gamma_A(R), \gamma_A(Q) > 1 \). This implies that \( (P,R) \) is not an intuitionistic fuzzy connected in \( A \). This implies that \( \mathcal{C}^A \) is not transitive with respect to an intuitionistic fuzzy set \( A \).

4. Intuitionistic fuzzy components related to an intuitionistic fuzzy digital topology

4.1. Plateau, Tops and bottom with respect to an intuitionistic fuzzy set

Definition 4.1. Let \( \mathbb{T} \) be any nonempty subset of \( \Sigma \). Then \( \mathbb{T} \) is connected with respect to an intuitionistic fuzzy set \( A \) in \( \Sigma \) if all \( P, Q \in \mathbb{T} \) are intuitionistic fuzzy connected in \( A \).

Definition 4.2. Let \( \mathbb{P} \) be any nonempty subset of \( \Sigma \) and let \( A \) be an intuitionistic fuzzy set in \( \Sigma \). Then \( \mathbb{P} \) is said to be an intuitionistic fuzzy plateau in \( A \) if it satisfies the following conditions:

(i) \( \mathbb{P} \) is intuitionistic fuzzy connected in \( A \)
(ii) \( \mu_A(P) = \mu_A(Q) \) and \( \gamma_A(P) \neq \gamma_A(Q) \), for all \( P, Q \in \mathbb{P} \)
(iii) \( \mu_A(P) \neq \mu_A(Q) \) and \( \gamma_A(P) = \gamma_A(Q) \), for all pairs of neighboring points \( P \in \mathbb{P}, Q \not\in \mathbb{P} \).

Clearly, any \( P \in \Sigma \), belongs to exactly one intuitionistic fuzzy plateau in \( A \).

Definition 4.3. Let \( \mathbb{P} \) be any nonempty subset of \( \Sigma \) and let \( A \) be an intuitionistic fuzzy set in \( \Sigma \). Then an intuitionistic fuzzy plateau \( \mathbb{P} \) in \( A \) is said to be an intuitionistic fuzzy top in \( A \) if it satisfies the following conditions:

(i) \( \mathbb{P} \) is intuitionistic fuzzy connected in \( A \)
(ii) \( \mu_A(P) > \mu_A(Q) \) and \( \gamma_A(P) \leq \gamma_A(Q) \), for all pairs of neighboring points \( P \in \mathbb{P}, Q \not\in \mathbb{P} \).

Otherwise it is said to be an intuitionistic fuzzy bottom in \( A \).

Definition 4.4. Let \( \mathbb{P} \) be any intuitionistic fuzzy top in \( A \). Now, some sets associated with an intuitionistic fuzzy top \( \mathbb{P} \) in \( A \) are defined as follows:

\[ \mathcal{L}_{\mathbb{P}} = \{ P \in \Sigma \left| \exists \text{ a path } \rho : P = P_0, P_1, \ldots, P_n = Q \in \mathbb{P} \text{ such that } \mu_A(P_{i+1}) \leq \mu_A(P_i) \text{ and } \gamma_A(P_{i+1}) > \gamma_A(P_i), 1 \leq i \leq n \} \]
\[ \forall \Pi = \{ P \in \Sigma | \text{there exists a path } \rho : P = P_0, P_1, \ldots, P_n = Q \in \Pi \text{ such that } \mu_A(P) \leq \mu_A(P) \leq \mu_A(Q) \text{ and } \gamma_A(P) > \gamma_A(P), 1 \leq i \leq n \}. \]

\[ \forall \Pi = \{ P \in \Sigma | \text{there exists a path } \rho : P = P_0, P_1, \ldots, P_n = Q \in \Pi \text{ such that } \mu_A(P) \leq \mu_A(P) \text{ and } \gamma_A(P) > \gamma_A(P), 1 \leq i \leq n \}. \]

**Remark 4.1.** For any intuitionistic fuzzy top in \( A, \Pi \subseteq \Pi_n \subseteq \forall \Pi \subseteq \forall \Pi_n. \)**

**Proof.** It is obvious.

**Note 4.1.** Let \( \Pi \) be any intuitionistic fuzzy top in \( A \). If \( P \in \forall \Pi_n \) and \( \mu_A(P) \geq \mu_A(\Pi) \) and \( \gamma_A(P) < \gamma_A(\Pi) \). Then \( P \in \Pi. \)

### 4.2. Connectedness in an intuitionistic fuzzy tops \( \Pi \)

**Proposition 4.1.** Let \( \Sigma \) be a rectangular array of integer-coordinate points. Let \( A \) be any intuitionistic fuzzy set in \( \Sigma \) and let \( \Pi \) be an intuitionistic fuzzy top. If \( P \in \forall \Pi \) and \( P \notin \Pi \). Then \( \mu_A(P) < \mu_A(\Pi) \) and \( \gamma_A(P) > \gamma_A(\Pi). \)

**Proof.** Assume that if \( P \in \forall \Pi \) and \( P \notin \Pi \) such that \( \mu_A(P) \geq \mu_A(\Pi) \) and \( \gamma_A(P) < \gamma_A(\Pi) \). This implies that there exists a path \( \rho \) from \( P \) to \( \Pi \) such that \( \mu_A(P) \geq \mu_A(P) \) and \( \gamma_A(P) < \gamma_A(P) < \gamma_A(\Pi) \), for all \( P \) on \( \rho \). But if \( P \notin \Pi \), then \( \rho \) must pass through a point \( Q \) that is adjacent to \( \Pi \) but not in \( \Pi \). Hence, \( \mu_A(Q) < \mu_A(\Pi) \) and \( \gamma_A(Q) \geq \gamma_A(\Pi) \), for any such \( Q \) with is a contradiction. Therefore, \( \mu_A(P) < \mu_A(\Pi) \) and \( \gamma_A(P) > \gamma_A(\Pi). \)

**Proposition 4.2.** Let \( \Sigma \) be a rectangular array of integer-coordinate points. Let \( A \) be any intuitionistic fuzzy set in \( \Sigma \) and let \( \Pi \) be an intuitionistic fuzzy top. Then \( \forall \Pi_n \) is the set of points of \( \Sigma \) that are intuitionistic fuzzy connected to points in \( \Pi \).

**Proof.** Let \( P \) be an intuitionistic fuzzy connected to \( Q \in \Pi \) with respect to an intuitionistic fuzzy set \( A \). Then by Proposition 3.3, there exists a path \( \rho \) from \( P \) to \( Q \) such that \( \mu_A(P) \geq \min(\mu_A(P), \mu_A(Q)) \) and \( \gamma_A(P) \leq \max(\gamma_A(P), \gamma_A(Q)) \), for \( 0 \leq i \leq n \) and for all \( P_i \) on \( \rho \). If \( \mu_A(P) > \mu_A(Q) \) and \( \gamma_A(P) \leq \gamma_A(Q) \) and \( P \notin \Pi \) implies that \( \mu_A(P) \geq \mu_A(Q) \) and \( \gamma_A(P) \leq \gamma_A(Q) \), for all \( P_i \) on \( \rho \). But by Proposition 4.1, it is impossible, since \( \rho \) must pass through a point \( Q' \) adjacent to \( \Pi \) but not in \( \Pi \) and for such a point, \( \mu_A(Q') < \mu_A(\Pi) \) and \( \gamma_A(Q') \geq \gamma_A(\Pi) \). Therefore, \( \mu_A(P) \leq \mu_A(Q) \) and \( \mu_A(P) \geq \mu_A(Q) \) and \( \gamma_A(P) \geq \gamma_A(Q) \) and \( \gamma_A(P) \geq \gamma_A(Q) \), for all \( P \) on \( \rho \). This implies that \( P \in \forall \Pi_n \).

Conversely, let \( P \in \forall \Pi \). Then \( \mu_A(P) \leq \mu_A(\Pi) \) and \( \gamma_A(P) \geq \gamma_A(\Pi) \). Then by Note 4.1, \( P \notin \Pi \). Again by Proposition 4.1., there exists a path \( \rho \) from \( P \) to a point.
Q of Π such that \( \mu_A(P) \geq \mu_A(Q) = \min(\mu_A(P), \mu_A(Q)) \) and \( \gamma_A(P) \leq \gamma_A(Q) = \max(\gamma_A(P), \gamma_A(Q)) \), for all \( P \) on \( \rho \). By Proposition 3.3, \( P \) is intuitionistic fuzzy connected to \( Q \).

**Proposition 4.3.** For any \( P \) in \( \Sigma \), there exists an intuitionistic fuzzy top such that \( P \in \mathcal{F}_\Pi \).

**Proof.** Let \( P \) be in intuitionistic fuzzy plateau \( \Pi_0 \). Suppose that if \( \Pi_0 \) is an intuitionistic fuzzy top. Then by Remark 4.1., \( P \in \Pi_0 \subseteq \mathcal{F}_\Pi \). If not, let \( P_1 \) be a neighbor of \( P_0 \in \Pi_0 \) such that \( \mu_A(P_1) > \mu_A(P_0) \) and \( \gamma_A(P_1) \leq \gamma_A(P_0) \). Thus a monotonically non-decreasing path \( \rho \) from \( P_0 \) to \( P_1 \) (going through \( \Pi_0 \) up to a neighbor of \( P_1 \). Repeat this argument with \( P_1 \) replacing \( P \) and continue in this way to obtain \( P_0, P_1, \ldots \). This must terminate at \( P_n \), since \( \Sigma \) is finite. Then \( \Pi_n \) is an intuitionistic fuzzy top. This implies that for a monotonic non-decreasing path \( \rho \) from \( P \) to \( P_n \). By Remark 4.1, \( P \in \mathcal{F}_\Pi \).

**Proposition 4.4.** Let \( \Pi \) and \( \Pi' \) be any two distinct intuitionistic fuzzy tops in \( \Sigma \). Then \( \Pi' \cap \mathcal{V}_\Pi = \emptyset \).

**Proof.** Suppose that \( P \in \Pi' \cap \mathcal{V}_\Pi \). Then there exists a path \( \rho \) from \( P \) to \( \Pi \) such that \( \mu_A(P) \leq \mu_A(P) \) and \( \gamma_A(P) > \gamma_A(P) \), for all \( P \) on \( \rho \). But for a point \( P_i \) adjacent to \( \Pi' \) but not in \( \Pi \), \( \mu_A(P) < \mu_A(\Pi') = \mu_A(P) \) and \( \gamma_A(P) \geq \gamma_A(\Pi') = \gamma_A(P) \) which is a contradiction. Therefore, \( \Pi' \cap \mathcal{V}_\Pi = \emptyset \).

**Proposition 4.5.** Let \( P \) and \( Q \) be any two points in \( \Sigma \). Then \( P \) is intuitionistic fuzzy connected to \( Q \) if and only if there exists an intuitionistic fuzzy top \( \Pi \) such that \( P \) and \( Q \) are both in \( \mathcal{V}_\Pi \).

**Proof.** Suppose that if there exists an intuitionistic fuzzy top \( \Pi \) such that \( P \) and \( Q \) are in \( \mathcal{V}_\Pi \). Then there are two paths \( \rho_1, \rho_2 \) from \( P \) and \( Q \) respectively to \( \Pi \) such that \( \mu_A(P) \geq \mu_A(Q) \) and \( \gamma_A(P) \leq \gamma_A(Q) \), for all \( P \) on \( \rho \), and similarly, \( \mu_A(Q) \geq \mu_A(Q) \) and \( \gamma_A(Q) \leq \gamma_A(Q) \). This implies that there is a path \( \rho \rho_2^{-1} \) from \( P \) to \( Q \) such that for all points \( R \) on this path, \( \mu_A(R) \geq \min(\mu_A(P), \mu_A(Q)) \) and \( \gamma_A(R) \leq \max(\gamma_A(P), \gamma_A(Q)) \). By Proposition 3.3, \( P \) is intuitionistic fuzzy connected to \( Q \).

Conversely, let \( P \) be intuitionistic fuzzy connected to \( Q \) and let \( \mu_A(P) \leq \mu_A(Q) \) and \( \gamma_A(P) \geq \gamma_A(Q) \). Let \( \rho' \) be a monotonic non-decreasing path from \( Q \) to some intuitionistic fuzzy top \( \Pi \). By Proposition 4.3, \( Q \in \mathcal{F}_\Pi \subseteq \mathcal{V}_\Pi \). On the other hand, by Proposition 3.3, there is a path \( \rho \) from \( P \) to \( Q \) such that \( \mu_A(P) \geq \min(\mu_A(P), \mu_A(Q)) = \mu_A(P) \) and \( \gamma_A(P) \leq \max(\gamma_A(P), \gamma_A(Q)) = \gamma_A(P) \) for all \( P \) on \( \rho \). This
implies that for all \( iQ \) on \( r \), \( A_i \geq A \) and \( \gamma_i \leq \gamma \), thus there is a path \( p p' \) from \( P \) to \( \Pi \) which guarantees that \( P \in \mathbb{V}_{\Pi} \).

**Corollary 4.1.** Let \( \Sigma \) be any rectangular-interior points and let \( A \) be an intuitionistic fuzzy set in \( \Sigma \). Then \( \Sigma \) is intuitionistic fuzzy connected with respect to an intuitionistic fuzzy set \( A \) if and only if there exists a unique intuitionistic fuzzy top in \( A \).

**Proof.** The proof follows from the Proposition 4.4 and 4.5.

**Definition 4.5.** Let \( A \) be any intuitionistic fuzzy set in \( \Sigma \) and let \( \Pi \) be any intuitionistic fuzzy top in \( A \). Then intuitionistic fuzzy set \( \Pi \) of \( \Sigma \) whose

\[
\mu_\Pi(P) = \begin{cases} 
\mu_A(P)/\mu_\Pi(P), & \text{if } P \in \mathbb{V}_{\Pi}, \\
0, & \text{otherwise.}
\end{cases}
\]

and

\[
\gamma_\Pi(P) = \begin{cases} 
\gamma_A(P)/\gamma_A(P), & \text{if } P \in \mathbb{V}_{\Pi}, \\
1, & \text{otherwise.}
\end{cases}
\]

such that \( \mu_\Pi + \gamma_\Pi \leq 1 \).

**Note 4.2.** Clearly, by Proposition 4.1, any intuitionistic fuzzy set \( \Pi \) of \( \Sigma \) whose \( \mu_\Pi(P) = 1 \) and \( \gamma_\Pi(P) = 0 \) if and only if \( P \in \Pi \).

5. **Intuitionistic fuzzy surroundness**

**Definition 5.1.** Let \( A, B, \) and \( C \) be any three intuitionistic fuzzy sets in \( \Sigma \). Then \( B \) separates \( A \) from \( C \) if for all points \( P, R \in \Sigma \) and all paths \( \rho \) from \( P \) to \( R \), there exists a point \( Q \) on \( \rho \) such that \( \mu_B(Q) \geq \min(\mu_A(P), \mu_C(R)) \) and \( \gamma_B(Q) \leq \max(\gamma_A(P), \gamma_C(R)) \).

In particular, \( B \) surrounds \( A \) if it separates \( A \) from the border \( B \) of \( \Sigma \). That is, for all \( P \in \Sigma \) and all paths \( \rho \) from \( P \) to \( B \), there exists a point \( Q \) on \( \rho \) such that \( \mu_B(Q) \geq \mu_A(P) \) and \( \gamma_B(Q) \leq \gamma_A(P) \).

**Note 5.1.** Clearly, if \( R \notin B \), then \( \mu_C(P) = 0 \) and \( \gamma_C(P) = 1 \). Similarly, if \( R \in B \), then \( \mu_C(P) = 1 \) and \( \gamma_C(P) = 0 \). This implies that

\[
\mu_A(P) = \begin{cases} 
\min(\mu_A(P), \mu_C(R)), & \text{if } R \in B, \\
0, & \text{otherwise.}
\end{cases}
\]

and

\[
\gamma_A(P) = \begin{cases} 
\max(\gamma_A(P), \gamma_C(R)), & \text{if } R \in B, \\
1, & \text{otherwise.}
\end{cases}
\]

**Proposition 5.1.** Let \( A, B, \) and \( C \) be any three intuitionistic fuzzy sets in \( \Sigma \). Then

(i) **Reflexivity:** \( A \) surrounds \( A \)

(ii) **Transitivity:** If \( A \) surrounds \( B \) and \( B \) surrounds \( C \), then \( A \) surrounds \( C \)

(iii) **Antisymmetry:** If \( A \) surrounds \( B \) and \( B \) surrounds \( A \), then \( A \cap B \) surrounds both of them.
Proof. (i) It is obvious if we take \( P = Q \).

(ii) Suppose that \( A \) surrounds \( B \) and \( B \) surrounds \( C \). Then for any \( P \in \Sigma \) and any path \( \rho \) from \( P \) to \( B \), there is a point \( Q \) on \( \rho \) such that \( \mu_B(Q) \geq \mu_C(P) \) and \( \gamma_B(Q) \leq \gamma_C(P) \). Moreover, on the part of path \( \rho \) between \( Q \) and \( B \), there is a point \( R \) such that \( \mu_A(R) \geq \mu_B(Q) \) and \( \gamma_A(R) \leq \gamma_B(Q) \). This implies that, for any \( P \in \Sigma \) and any path \( \rho \) from \( P \) to \( B \), there is a point \( R \) such that \( \mu_A(R) \geq \mu_C(P) \) and \( \gamma_A(R) \leq \gamma_C(P) \). Therefore, \( A \) surrounds \( C \).

(iii) Let \( \rho \) be any path from \( P \) to \( B \) and let \( Q \) be the last point on \( \rho \) such that \( \mu_B(Q) \geq \mu_A(P) \) and \( \gamma_B(Q) \leq \gamma_A(P) \). Since \( A \) surrounds \( B \), there must a point \( Q' \) on \( \rho \) beyond (or possibly itself) \( Q \) such that \( \mu_A(Q') \geq \mu_B(Q) \) and \( \gamma_A(Q') \leq \gamma_B(Q) \). Since \( B \) surrounds \( A \), there must a point \( Q'' \) on \( \rho \) beyond (or equal to) \( Q \) such that \( \mu_B(Q'') \geq \mu_A(Q') \) and \( \gamma_B(Q'') \leq \gamma_A(Q') \). Suppose that \( Q = Q' = Q'' \), then \( \mu_A(Q) \lor \mu_A(Q) \geq \mu_A(P) \) and \( \gamma_A(Q) \land \gamma_A(Q) \leq \gamma_A(P) \). Since \( P \) is any arbitrary point. If \( \mu_A(Q) = \mu_B(Q) \) and \( \gamma_A(Q) = \gamma_B(Q) \), then the two intuitionistic fuzzy sets \( A \) and \( B \) are equal. Otherwise it is not equal. This implies that \( A \cap B \) surrounds \( A \) and \( B \) respectively. Respectively.

Note 5.2. Intuitionistic fuzzy surrounds is an weak partial order relation with respects to an intuitionistic fuzzy sets.

References

Generalization of An Intuitionistic Fuzzy $\mathcal{G}_{str}$ Open Sets in An Intuitionistic Fuzzy Grill Structure Spaces

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Abstract:
The purpose of this paper is to introduce the concepts of an intuitionistic fuzzy grill, intuitionistic fuzzy $\mathcal{G}$ structure space, intuitionistic fuzzy $\delta_{a_{\alpha}}$ set and intuitionistic fuzzy $\alpha_{a_{\alpha}}$ open set. The concepts of an intuitionistic fuzzy $E_{a_{\alpha}}$ continuous function, intuitionistic fuzzy $\alpha_{a_{\alpha}}$-$T_i$ space, $i = 0, 1, 2$ and intuitionistic fuzzy $\alpha_{a_{\alpha}}$-co-closed graphs are defined. Some interesting properties are established.

Keywords:
Intuitionistic fuzzy grill, intuitionistic fuzzy $\mathcal{G}$ structure space, intuitionistic fuzzy $\delta_{a_{\alpha}}$ set and intuitionistic fuzzy $\alpha_{a_{\alpha}}$ (resp. $\alpha_{a_{\alpha}}$, semi$_{a_{\alpha}}$, pre$_{a_{\alpha}}$, regular$_{a_{\alpha}}$ and $\beta_{a_{\alpha}}$) open set, intuitionistic fuzzy $\alpha_{a_{\alpha}}$ exterior, intuitionistic fuzzy $E_{a_{\alpha}}$ continuous function, intuitionistic fuzzy $\alpha_{a_{\alpha}}$-$T_i$ space, $i = 0, 1, 2$ and intuitionistic fuzzy $\alpha_{a_{\alpha}}$-co-closed graphs.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [7] and later Atanassov [1] generalized the idea to intuitionistic fuzzy sets. On the other hand, Coker [2] introduced the notions of an intuitionistic fuzzy topological spaces, intuitionistic fuzzy continuity and some other related concepts. The concept of an intuitionistic fuzzy $\alpha$-closed set was introduced by H. Gürçay and D. Coker [5]. The concept of fuzzy grill was introduced by Sumita Das, M. N. Mukherjee [6]. Erdal Ekici [4] studied slightly precontinuous functions, separation axioms and pre-co-closed graphs in fuzzy topological space. In this paper, the concepts of an intuitionistic fuzzy grill, intuitionistic fuzzy $\mathcal{G}$ structure space, intuitionistic fuzzy $\delta_{a_{\alpha}}$ set and intuitionistic fuzzy $\alpha_{a_{\alpha}}$ open set are introduced. Some interesting properties of separation axioms in intuitionistic...
fuzzy grill structure space with intuitionistic fuzzy continuous function are established.

2. Preliminaries

Definition 2.1 [1]. Let \( X \) be a nonempty fixed set and \( I \) be the closed interval \([0,1]\). An intuitionistic fuzzy set \((IFS)\) \( A \) is an object of the following form \( A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\} \), where the mappings \( \mu_A : X \rightarrow I \) and \( \gamma_A : X \rightarrow I \) denote the degree of membership (namely \( \mu_A(x) \)) and the degree of nonmembership (namely \( \gamma_A(x) \)) for each element \( x \in X \) to the set \( A \), respectively, and \( 0 \leq \mu_A(x) + \gamma_A(x) \leq 1 \) for each \( x \in X \). Obviously, every fuzzy set \( A \) on a nonempty set \( X \) is an IFS of the following form, \( A = \{(x, \mu_A(x), 1 - \mu_A(x)) : x \in X\} \). For the sake of simplicity, we shall use the symbol \( A = (x, \mu_A(x), \gamma_A(x)) \) for the intuitionistic fuzzy set \( A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\} \).

Definition 2.2 [1]. Let \( X \) be a nonempty set and the IFSs \( A \) and \( B \) in the form \( A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\} \), \( B = \{(x, \mu_B(x), \gamma_B(x)) : x \in X\} \). Then

(i) \( A \subseteq B \) iff \( \mu_A(x) \leq \mu_B(x) \) and \( \gamma_A(x) \geq \gamma_B(x) \) for all \( x \in X \);

(ii) \( \overline{A} = \{(x, \gamma_A(x), \mu_A(x)) : x \in X\} \);

(iii) \( A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \max(\gamma_A(x), \gamma_B(x))) : x \in X\} \);

(iv) \( A \cup B = \{(x, \max(\mu_A(x), \mu_B(x)), \min(\gamma_A(x), \gamma_B(x))) : x \in X\} \).

Definition 2.3 [1]. The IFSs \( 0_\ast \) and \( 1_\ast \) are defined by \( 0_\ast = \{(x, 0, 1) : x \in X\} \) and \( 1_\ast = \{(x, 1, 0) : x \in X\} \).

Definition 2.4 [2]. An intuitionistic fuzzy topology \((IFT)\) in Coker’s sense on a nonempty set \( X \) is a family \( \tau \) of IFSs in \( X \) satisfying the following axioms:

(i) \( 0_\ast, 1_\ast \in \tau \);

(ii) \( G_1 \cap G_2 \in \tau \) for any \( G_1, G_2 \in \tau \);

(iii) \( \bigcup \{G_i : i \in I\} \in \tau \).

In this case the ordered pair \((X, \tau)\) is called an intuitionistic fuzzy topological space \((IFTS)\) on \( X \) and each IFS in \( \tau \) is called an intuitionistic fuzzy open set \((IFOS)\). The complement \( \overline{A} \) of an IFOS \( A \) in \( X \) is called an intuitionistic fuzzy closed set \((IFCS)\) in \( X \).
Definition 2.5 [2]. Let \( A \) be an IFS in \( IFTS_X \). Then \( \text{int}(A) = \bigcup \{G|G \text{ is an IFOS in } X \} \) is called an intuitionistic fuzzy interior of \( A \); \( \text{cl}(A) = \bigcap \{G|G \text{ is an IFCS in } X \} \) is called an intuitionistic fuzzy closure of \( A \).

Proposition 2.1 [1]. For any IFS \( A \) in \( (X, \tau) \) we have

(i) \( \text{cl}(A) = \overline{\text{int}(A)} \)

(ii) \( \text{int}(A) = \overline{\text{cl}(A)} \)

Corollary 2.1 [2]. Let \( A, A_i (i \in I), B, B_j (j \in J) \) IFSs in \( Y \) and \( f: X \to Y \) a function. Then

(i) \( A \subseteq f^{-1}(f(A)) \) (If \( f \) is injective, then \( A = f^{-1}(f(A)) \)).

(ii) \( f(f^{-1}(B)) \subseteq B \) (If \( f \) is surjective, then \( f(f^{-1}(B)) = B \)).

(iii) \( f^{-1}(\bigcup B_j) = \bigcup f^{-1}(B_j) \).

(iv) \( f^{-1}(\bigcap B_j) = \bigcap f^{-1}(B_j) \).

(v) \( f^{-1}(1_c) = 1_c \).

(vi) \( f^{-1}(0_c) = 0_c \).

(vii) \( f^{-1}(\bar{B}) = \bar{f^{-1}(B)} \).

Definition 2.6 [3]. Let \( X \) be a nonempty set and \( x \in X \) a fixed element in \( X \). If \( r \in I_0, s \in I_1 \) are fixed real numbers such that \( r + s \leq 1 \), then the IFS \( x_{r,s} = \{x, r, 1-x_{r,s}\} \) is called an intuitionistic fuzzy point (IFP) in \( X \), where \( r \) denotes the degree of membership of \( x_{r,s} \), \( s \) denotes the degree of nonmembership of \( x_{r,s} \) and \( x \in X \) the support of \( x_{r,s} \). The IFP \( x_{r,s} \) is contained in the IFS \( A(x_{r,s} \in A) \) if and only if \( r < \mu_A(x) \), \( s > \gamma_A(x) \).

Definition 2.7 [4]. An IFSs \( A \) and \( B \) are said to be quasi coincident with the IFSA, denoted by \( AqB \) if and only if there exists an element \( x \in X \) such that \( \mu_A(x) > \gamma_B(x) \) or \( \gamma_A(x) < \mu_B(x) \). If \( A \) is not quasi coincident with \( B \), denoted \( A\tilde{q}B \).

Definition 2.8 [5]. Let \( A \) be an IFS of an IFTS \( X \). Then \( A \) is called an intuitionistic fuzzy \( \alpha \)-open set (IF\( \alpha \)OS) if \( A \subseteq \text{int}(\overline{\text{cl}(A)}) \). The complement of an intuitionistic fuzzy \( \alpha \)-open set is called an intuitionistic fuzzy \( \alpha \)-closed set (IF\( \alpha \)CS).
3. Intuitionistic fuzzy operators with respect to an intuitionistic fuzzy grills

**Definition 3.1.** Let $\xi^X$ be the collection of all intuitionistic fuzzy sets in $X$. A collection $\mathcal{G} \subseteq \xi^X$ is said to be an intuitionistic fuzzy stack on $X$ if $A \subseteq B$ and $A \in \mathcal{G}$ then $B \in \mathcal{G}$.

**Definition 3.2.** Let $\xi^X$ be the collection of all intuitionistic fuzzy sets in $X$. An intuitionistic fuzzy grill $\mathcal{H}$ on $X$ is an intuitionistic fuzzy stack on $X$ if $\mathcal{H}$ satisfies the following conditions:

(i) $\emptyset \notin \mathcal{H}$,
(ii) If $A, B \in \xi^X$ and $A \cup B \in \mathcal{H}$, then $A \in \mathcal{H}$ (or) $B \in \mathcal{H}$.

**Definition 3.3.** Let $(X, T)$ be an intuitionistic fuzzy topological space and let $\xi^X$ be the collection of all intuitionistic fuzzy sets in $X$. Let $\mathcal{H}$ be an intuitionistic fuzzy grill on $X$. A function $\Phi_\mathcal{H} : \xi^X \to \xi^X$ is defined by

$$\Phi_\mathcal{H}(A) = \bigcup \{I F \text{int}(\overline{A}) \cap \overline{U} | A \cap U \in \mathcal{H}, U \in T \}$$

for each $A \in \xi^X$. The function $\Phi_\mathcal{H}$ is an intuitionistic fuzzy operator associated with an intuitionistic fuzzy grill $\mathcal{H}$ and an intuitionistic fuzzy topology $T$.

**Remark 3.1.** Let $(X, T)$ be an intuitionistic fuzzy topological space. Let $\Phi_\mathcal{H}$ be an intuitionistic fuzzy operator associated with an intuitionistic fuzzy grill $\mathcal{H}$ and an intuitionistic fuzzy topology $T$. Then

(i) $\Phi_\mathcal{H}(\emptyset) = \emptyset = \Phi_\mathcal{H}(1)$
(ii) If $A, B \in \xi^X$ and $A \subseteq B$, then $\Phi_\mathcal{H}(A) \subseteq \Phi_\mathcal{H}(B)$

**Proof.** The proof of (i) and (ii) are follows from the definition of $\Phi_\mathcal{H}$.

**Definition 3.4.** Let $(X, T)$ be an intuitionistic fuzzy topological space and let $\mathcal{G}$ be an intuitionistic fuzzy grill on $X$. Let $\Phi_\mathcal{G}$ be an intuitionistic fuzzy operator associated with an intuitionistic fuzzy grill $\mathcal{G}$ and an intuitionistic fuzzy topology $T$. A function $\Psi_\mathcal{G} : T \to \xi^X$ is defined by $\Psi_\mathcal{G}(A) = A \cup IF \text{int}(\overline{\Phi_\mathcal{G}})$ for each $A \in T$. The function $\Psi_\mathcal{G}$ is an intuitionistic fuzzy operator associated with $\Phi_\mathcal{G}$.

**Definition 3.5.** Let $(X, T)$ be an intuitionistic fuzzy topological space and let $\mathcal{G}$ be an intuitionistic fuzzy grill on $X$. Let $\Psi_\mathcal{G}$ be an intuitionistic fuzzy operator associated with $\Phi_\mathcal{G}$. A collection $\mathcal{G}_{str} = \{A | \Psi_\mathcal{G}(A) = A \} \cup \{1\}$ is said to be an intuitionistic fuzzy $\mathcal{G}$ structure on $X$. Then $(X, \mathcal{G}_{str})$ is said to be an intuitionistic fuzzy $\mathcal{G}$...
structure space. Every member of $\mathcal{G}_{str}$ is an intuitionistic fuzzy $\mathcal{G}_{str}$ open set (in short, $IF\mathcal{G}_{str}OS$) and the complement of an intuitionistic fuzzy $\mathcal{G}_{str}$ open set is an intuitionistic fuzzy $\mathcal{G}_{str}$ closed set (in short, $IF\mathcal{G}_{str}CS$).

**Definition 3.6.** Let $(X, \mathcal{G}_{str})$ be an intuitionistic fuzzy $\mathcal{G}$ structure space and let $A \in \xi^X$. Then the
(i) intuitionistic fuzzy $\mathcal{G}_{str}$ closure of $A$ is denoted and defined by
$$IF_{\mathcal{G}_{str}}\text{cl}(A) = \bigcap \left\{ B \in \xi^X \mid B \supseteq A \text{ and } \overline{B} \in \mathcal{G}_{str} \right\}$$
(ii) intuitionistic fuzzy $\mathcal{G}_{str}$ interior of $A$ is denoted and defined by
$$IF_{\mathcal{G}_{str}}\text{int}(A) = \bigcup \left\{ B \in \xi^X \mid B \subseteq A \text{ and } B \in \mathcal{G}_{str} \right\}$$

**Remark 3.2.** Let $(X, \mathcal{G}_{str})$ be an intuitionistic fuzzy $\mathcal{G}$ structure space. For any $A, B \in \xi^X$,
(i) $IF_{\mathcal{G}_{str}}\text{cl}(A) = A$ if and only if $A$ is an intuitionistic fuzzy $\mathcal{G}_{str}$ closed set.
(ii) $IF_{\mathcal{G}_{str}}\text{int}(A) = A$ if and only if $A$ is an intuitionistic fuzzy $\mathcal{G}_{str}$ open set.
(iii) $IF_{\mathcal{G}_{str}}\text{int}(A) \subseteq A \subseteq IF_{\mathcal{G}_{str}}\text{cl}(A)$.
(iv) $IF_{\mathcal{G}_{str}}\text{int}(\emptyset) = \emptyset$ and $IF_{\mathcal{G}_{str}}\text{int}(X) = X$.
(v) $IF_{\mathcal{G}_{str}}\text{int}(\overline{A}) = IF_{\mathcal{G}_{str}}\text{cl}(A)$ and $IF_{\mathcal{G}_{str}}\text{cl}(\overline{A}) = IF_{\mathcal{G}_{str}}\text{int}(A)$.

4. Properties of an intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ set and $\alpha_{\mathcal{G}_{str}}$ open set in an intuitionistic fuzzy $\mathcal{G}$ structure spaces

**Definition 4.1.** Let $(X, \mathcal{G}_{str})$ be an intuitionistic fuzzy $\mathcal{G}$ structure space and let $A \in \xi^X$. Then $A$ is said to be an intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ set (in short, $IF\delta_{\mathcal{G}_{str}}S$) if
$$IF_{\mathcal{G}_{str}}\text{int}\left(IF_{\mathcal{G}_{str}}\text{cl}(A)\right) \subseteq IF_{\mathcal{G}_{str}}\text{cl}\left(IF_{\mathcal{G}_{str}}\text{int}(A)\right).$$

**Proposition 4.1.** Let $(X, \mathcal{G}_{str})$ be an intuitionistic fuzzy $\mathcal{G}$ structure space. Then
(i) The complement of an intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ set is an intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ set.
(ii) Finite union of an intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ sets is an intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ set.
(iii) Finite intersection of an intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ sets is an intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ set.
(iv) Every intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ sets is an intuitionistic fuzzy $\mathcal{G}_{str}$ open set.
Proof: (i) Let \( A \) be an intuitionistic fuzzy \( \delta_{\mathcal{G}} \) set. Then \( IF_{\mathcal{G}} \text{ int}(IF_{\mathcal{G}} \text{ cl}(A)) \subseteq IF_{\mathcal{G}} \text{ cl}(IF_{\mathcal{G}} \text{ int}(A)). \) Taking complement on both sides, we have \( IF_{\mathcal{G}} \text{ int}(IF_{\mathcal{G}} \text{ cl}(A)) \supseteq IF_{\mathcal{G}} \text{ cl}(IF_{\mathcal{G}} \text{ int}(A)) \) \( \supseteq IF_{\mathcal{G}} \text{ cl}(IF_{\mathcal{G}} \text{ int}(A)) \) \( \supseteq IF_{\mathcal{G}} \text{ cl}(IF_{\mathcal{G}} \text{ int}(\overline{A})). \) Hence \( \overline{A} \) is an intuitionistic fuzzy \( \delta_{\mathcal{G}} \) set.

(ii) Let \( A \) and \( B \) be any two intuitionistic fuzzy \( \delta_{\mathcal{G}} \) sets. Then

\[
\begin{align*}
IF_{\mathcal{G}} \text{ int}(IF_{\mathcal{G}} \text{ cl}(A)) & \subseteq IF_{\mathcal{G}} \text{ cl}(IF_{\mathcal{G}} \text{ int}(A)) \\
IF_{\mathcal{G}} \text{ int}(IF_{\mathcal{G}} \text{ cl}(B)) & \subseteq IF_{\mathcal{G}} \text{ cl}(IF_{\mathcal{G}} \text{ int}(B))
\end{align*}
\]

Now, taking union of 3.1 and 3.2, we have \( IF_{\mathcal{G}} \text{ int}(IF_{\mathcal{G}} \text{ cl}(A)) \cup IF_{\mathcal{G}} \text{ int}(IF_{\mathcal{G}} \text{ cl}(B)) \subseteq IF_{\mathcal{G}} \text{ cl}(IF_{\mathcal{G}} \text{ int}(A) \cup IF_{\mathcal{G}} \text{ int}(B)) \) \( IF_{\mathcal{G}} \text{ int}(IF_{\mathcal{G}} \text{ cl}(A \cup B)) \subseteq IF_{\mathcal{G}} \text{ cl}(IF_{\mathcal{G}} \text{ int}(A \cup B)). \) Hence \( A \cap B \) is an intuitionistic fuzzy \( \delta_{\mathcal{G}} \) set.

(iii) The proof is obvious by taking complement of (ii).

(iv) It is obvious.

**Note 4.1.** In general, arbitrary union of an intuitionistic fuzzy \( \delta_{\mathcal{G}} \) set need not be intuitionistic fuzzy \( \delta_{\mathcal{G}} \) set.

**Definition 4.2.** Let \( (X, \mathcal{G}) \) be an intuitionistic fuzzy \( \mathcal{G} \) structure space and let \( A \in \mathcal{G}^X \). Then \( A \) is said to be an

(i) intuitionistic fuzzy \( \text{semi}_{\mathcal{G}} \) open set (in short, \( \text{IFS}_{\mathcal{G}} \text{ OS} \)) if \( A \subseteq IF_{\mathcal{G}} \text{ cl}(IF_{\mathcal{G}} \text{ int}(A)). \) The complement of an intuitionistic fuzzy \( \text{semi}_{\mathcal{G}} \) open set is said to be an intuitionistic fuzzy \( \text{semi}_{\mathcal{G}} \) closed set (in short, \( \text{IFS}_{\mathcal{G}} \text{ CS} \)).

(ii) intuitionistic fuzzy \( \text{pre}_{\mathcal{G}} \) open set (in short, \( \text{IFP}_{\mathcal{G}} \text{ OS} \)) if \( A \subseteq IF_{\mathcal{G}} \text{ int}(IF_{\mathcal{G}} \text{ cl}(A)). \) The complement of an intuitionistic fuzzy \( \text{pre}_{\mathcal{G}} \) open set is said to be an intuitionistic fuzzy \( \text{pre}_{\mathcal{G}} \) closed set (in short, \( \text{IFP}_{\mathcal{G}} \text{ CS} \)).

(iii) intuitionistic fuzzy \( \alpha_{\mathcal{G}} \) open set (in short, \( \text{IF} \alpha_{\mathcal{G}} \text{ OS} \)) if

\[
A \subseteq IF_{\mathcal{G}} \text{ int}(IF_{\mathcal{G}} \text{ cl}(IF_{\mathcal{G}} \text{ int}(A))).
\]

The complement of an intuitionistic fuzzy \( \alpha_{\mathcal{G}} \) open set is said to be an intuitionistic fuzzy \( \alpha_{\mathcal{G}} \) closed set (in short, \( \text{IF} \alpha_{\mathcal{G}} \text{ CS} \)).

(iv) intuitionistic fuzzy \( \beta_{\mathcal{G}} \) open set (in short, \( \text{IF} \beta_{\mathcal{G}} \text{ OS} \)) if
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\[ A \subseteq IF_{\alpha_{\mathcal{Q}_m}} cl \left( IF_{\beta_{\mathcal{Q}_m}} \text{int} \left( IF_{\beta_{\mathcal{Q}_m}} cl(A) \right) \right). \]

The complement of an intuitionistic fuzzy \( \beta_{\mathcal{Q}_m} \) open set is said to be an intuitionistic fuzzy \( \alpha_{\mathcal{Q}_m} \) closed set (in short, \( IF_{\alpha_{\mathcal{Q}_m}} \text{CS} \)).

(v) intuitionistic fuzzy regular \( \beta_{\mathcal{Q}_m} \) open set (in short, \( IF_{\beta_{\mathcal{Q}_m}} \text{OS} \)) if

\[ A = IF_{\beta_{\mathcal{Q}_m}} \text{int} \left( IF_{\beta_{\mathcal{Q}_m}} cl(A) \right). \]

The complement of an intuitionistic fuzzy regular \( \beta_{\mathcal{Q}_m} \) open set is said to be an intuitionistic fuzzy regular \( \beta_{\mathcal{Q}_m} \) closed set (in short, \( IF_{\beta_{\mathcal{Q}_m}} \text{CS} \)).

**Note. 4.2.** The family of all intuitionistic fuzzy semi \( \beta_{\mathcal{Q}_m} \), (resp. pre \( \beta_{\mathcal{Q}_m} \), \( \alpha_{\mathcal{Q}_m} \), \( \beta_{\mathcal{Q}_m} \) and regular \( \beta_{\mathcal{Q}_m} \) ) open sets are denoted by \( IF_{\beta_{\mathcal{Q}_m}} O(X) \) (resp. \( IF_{\beta_{\mathcal{Q}_m}} O(X) \), \( IF_{\alpha_{\mathcal{Q}_m}} O(X) \), \( IF_{\beta_{\mathcal{Q}_m}} O(X) \) and \( IF_{\beta_{\mathcal{Q}_m}} O(X) \)).

**Definition 4.3.** Let \( (X, \mathcal{G}_{\mathcal{Q}_m}) \) be an intuitionistic fuzzy \( \mathcal{G} \) structure space and let \( A \in \xi^X \). Then the

(i) intuitionistic fuzzy \( \alpha_{\mathcal{Q}_m} \) closure of \( A \) is denoted and defined by

\[ IF_{\alpha_{\mathcal{Q}_m}} cl(A) = \bigcap \left\{ B \in \xi^X \left| B \supseteq A \text{ and } B \in IF_{\alpha_{\mathcal{Q}_m}} O(X) \right. \right\} \]

(ii) intuitionistic fuzzy \( \alpha_{\mathcal{Q}_m} \) interior of \( A \) is denoted and defined by

\[ IF_{\alpha_{\mathcal{Q}_m}} \text{int} (A) = \bigcup \left\{ B \in \xi^X \left| B \subseteq A \text{ and } B \in IF_{\alpha_{\mathcal{Q}_m}} O(X) \right. \right\} \]

**Remark 4.1.** Let \( (X, \mathcal{G}_{\mathcal{Q}_m}) \) be an intuitionistic fuzzy \( \mathcal{G} \) structure space. For any \( A, B \in \xi^X \),

(i) \( IF_{\alpha_{\mathcal{Q}_m}} cl(A) = A \) if and only if \( A \) is an intuitionistic fuzzy \( \alpha_{\mathcal{Q}_m} \) closed set.

(ii) \( IF_{\alpha_{\mathcal{Q}_m}} \text{int} (A) = A \) if and only if \( A \) is an intuitionistic fuzzy \( \alpha_{\mathcal{Q}_m} \) open set.

(iii) \( IF_{\alpha_{\mathcal{Q}_m}} \text{int} (A) \subseteq A \subseteq IF_{\alpha_{\mathcal{Q}_m}} cl(A) \).

(iv) \( IF_{\alpha_{\mathcal{Q}_m}} \text{int} (1.) = 1. = IF_{\alpha_{\mathcal{Q}_m}} cl(1.) \) and \( IF_{\alpha_{\mathcal{Q}_m}} \text{int} (0.) = 0. = IF_{\alpha_{\mathcal{Q}_m}} cl(0.) \).

(v) \( IF_{\alpha_{\mathcal{Q}_m}} \text{int} (\overline{A}) = IF_{\alpha_{\mathcal{Q}_m}} cl(\overline{A}) \) and \( IF_{\alpha_{\mathcal{Q}_m}} cl(\overline{A}) = IF_{\alpha_{\mathcal{Q}_m}} \text{int} (A) \).

**Proposition 4.2.** Let \( (X, \mathcal{G}_{\mathcal{Q}_m}) \) be an intuitionistic fuzzy \( \mathcal{G} \) structure space. Then

(i) Every intuitionistic fuzzy regular \( \beta_{\mathcal{Q}_m} \) open set is an intuitionistic fuzzy \( \beta_{\mathcal{Q}_m} \) open set.
(ii) Every intuitionistic fuzzy intuitionistic fuzzy \( 'G_{str} \) open set is an intuitionistic fuzzy semi\( _{str} \) set (resp. pre\( _{str} \), \( \alpha_{str} \) and \( \beta_{str} \)) open set.

**Proof.** It is obvious.

**Remark 4.2.** The converse of the Proposition 4.2 need not be true as shown in Example 4.1.

**Example 4.1.** Let \( X = \{a, b\} \) be a nonempty set. Let \( A = \{x, (a/0.3, b/0.5), (a/0.2, b/0.3)\} \), \( B = \{x, (a/0.3, b/0.4), (a/0.2, b/0.4)\} \), \( C = \{x, (a/0.2, b/0.2), (a/0.7, b/0.5)\} \) and \( E = \{x, (a/0.7, b/0.8), (a/0.2, b/0.2)\} \) be intuitionistic fuzzy sets on \( X \). The family \( T = \{0, 1, A, B, C, D\} \) is an intuitionistic fuzzy topology on \( X \) and the family \( \mathcal{G} = \{G \in \mathcal{X}^X \mid 0.2 \leq \mu_G(x) \leq 1 \text{ and } 0 \leq \gamma_G(x) \leq 0.8\} \) is an intuitionistic fuzzy grill on \( X \). Then the family \( \mathcal{G}_{str} = \{0, 1, A\} \) is an intuitionistic fuzzy \( \mathcal{G} \) structure on \( X \). Therefore, \( (X, \mathcal{G}_{str}) \) is an intuitionistic fuzzy \( \mathcal{G} \) structure space. Now,

(i) \( F = \{x, (a/0.3, b/0.5), (a/0.2, b/0.3)\} \) is an intuitionistic fuzzy \( \mathcal{G}_{str} \) open set but need not be an intuitionistic fuzzy regular\( _{str} \) open set in \( X \).

(ii) \( H = \{x, (a/0.3, b/0.6), (a/0.2, b/0.3)\} \) is an intuitionistic fuzzy \( \alpha_{str} \) (resp. semi\( _{str} \)) open set but need not be an intuitionistic fuzzy \( \mathcal{G}_{str} \) open set in \( X \).

(iii) \( K = \{x, (a/0.3, b/0.7), (a/0.2, b/0.3)\} \) is an intuitionistic fuzzy pre\( _{str} \) (resp. \( \beta_{str} \)) open set but need not be an intuitionistic fuzzy \( \mathcal{G}_{str} \) open set in \( X \).

**Proposition 4.3.** Every intuitionistic fuzzy regular\( _{str} \) open set is an intuitionistic fuzzy \( \delta_{str} \) set.

**Proof.** Let \( A \) be an intuitionistic fuzzy regular\( _{str} \) open set. Then \( IF_{str} \int \{IF_{str} \text{ cl}(A)\} = A \subseteq IF_{str} \text{ cl}(A) \). Since every intuitionistic fuzzy regular\( _{str} \) open set is an intuitionistic fuzzy \( \mathcal{G}_{str} \) open set, \( IF_{str} \int \{IF_{str} \text{ cl}(A)\} \subseteq IF_{str} \text{ cl}(IF_{str} \int(A)) \). Hence \( A \) is an intuitionistic fuzzy \( \delta_{str} \) set.

**Remark 4.3.** The converse of the Proposition 4.3 need not be true as shown in Example 4.2.

**Example 4.2.** Let \( X = \{a, b\} \) be a nonempty set. Let \( A = \{x, (a/0.3, b/0.5), (a/0.2, b/0.3)\} \), \( B = \{x, (a/0.3, b/0.4), (a/0.2, b/0.4)\} \), \( C = \{x, (a/0.2, b/0.2), (a/0.7, b/0.5)\} \) and \( E = \{x, (a/0.7, b/0.8), (a/0.2, b/0.2)\} \) be intuitionistic fuzzy sets on \( X \). The family \( T = \{0, 1, A, B, C, D\} \) is an intuitionistic fuzzy topology on \( X \) and the family \( \mathcal{G} = \{G \in \mathcal{X}^X \mid 0.2 \leq \mu_G(x) \leq 1 \text{ and } 0 \leq \gamma_G(x) \leq 0.8\} \) is an intuitionistic fuzzy grill on \( X \).
Then the family \( \mathcal{G}_{\alpha} = \{0, 1, A\} \) is an intuitionistic fuzzy \( \mathcal{G} \) structure on \( X \). Therefore, \((X, \mathcal{G}_{\alpha})\) is an intuitionistic fuzzy \( \mathcal{G} \) structure space. Now, \( F = (x,(a/0.3,b/0.5), (a/0.2,b/0.2)) \) is an intuitionistic fuzzy \( \delta_{\alpha} \) set but need not be an intuitionistic fuzzy regular\( g_{\alpha} \) open set in \( X \).

**Proposition 4.4.** Every intuitionistic fuzzy \( \alpha_{\alpha} \) open set is an intuitionistic fuzzy semi\( \alpha_{\alpha} \) open set.

**Proof.** Let \( A \) be an intuitionistic fuzzy \( \alpha_{\alpha} \) open set. Then

\[
A \subseteq IF_{\alpha_{\alpha}} \text{ int}(IF_{\alpha_{\alpha}} cl(IF_{\alpha_{\alpha}} \text{ int}(A))) \subseteq IF_{\alpha_{\alpha}} \text{ cl}(IF_{\alpha_{\alpha}} \text{ int}(A))
\]

Hence \( A \) is an intuitionistic fuzzy semi\( \alpha_{\alpha} \) open set.

**Remark 4.4.** The converse of the Proposition 4.4 need not be true as shown in Example 4.3.

**Proposition 4.5.** Every intuitionistic fuzzy pre\( \alpha_{\alpha} \) open set is an intuitionistic fuzzy \( \beta_{\alpha} \) open set.

**Proof.** Let \( A \) be an intuitionistic fuzzy pre\( \alpha_{\alpha} \) set. Then

\[
A \subseteq IF_{\alpha_{\alpha}} \text{ int}(IF_{\alpha_{\alpha}} cl(IF_{\alpha_{\alpha}} \text{ cl}(A))) \subseteq IF_{\alpha_{\alpha}} \text{ cl}(IF_{\alpha_{\alpha}} \text{ int}(A))
\]

Hence \( A \) is an intuitionistic fuzzy \( \beta_{\alpha} \) open set.

**Remark 4.5.** The converse of the Proposition 4.5 need not be true as shown in Example 4.3.

**Example 4.3.** Let \( X = \{a,b\} \) be a nonempty set. Let

\[
A_0 = \{x,(a/0.2,b/0.3),(a/0.7,b/0.7)\}, \quad A_1 = \{x,(a/0.4,b/0.6),(a/0.2,b/0.4)\}, \\
A_2 = \{x,(a/0.7,b/0.6),(a/0.3,b/0.4)\}, \quad A_3 = \{x,(a/0.1,b/0.8),(a/0.9,b/0.2)\}, \\
A_4 = \{x,(a/0.1,b/0.3),(a/0.9,b/0.7)\}, \quad A_5 = \{x,(a/0.7,b/0.8),(a/0.3,b/0.2)\}, \\
A_6 = \{x,(a/0.7,b/0.6),(a/0.2,b/0.4)\}, \quad A_7 = \{x,(a/0.7,b/0.8),(a/0.2,b/0.2)\}, \\
A_8 = \{x,(a/0.4,b/0.8),(a/0.2,b/0.2)\}, \quad A_9 = \{x,(a/0.2,b/0.8),(a/0.7,b/0.2)\}, \\
A_{10} = \{x,(a/0.1,b/0.6),(a/0.9,b/0.4)\}, \quad A_{11} = \{x,(a/0.4,b/0.8),(a/0.3,b/0.2)\}, \\
A_{12} = \{x,(a/0.4,b/0.6),(a/0.3,b/0.4)\}
\]

be intuitionistic fuzzy sets on \( X \).

The family \( T = \{0, 1, A_i \}_{i=1,2, \cdots, 13} \) is an intuitionistic fuzzy topology on \( X \) and the family \( \mathcal{G} = \{G \in \mathcal{G}^\times \mid 0.1 \leq \mu_G(x) \leq 1 \text{ and } 0 \leq \nu_G(x) \leq 0.9\} \) is an intuitionistic fuzzy...
grill on \( X \). Then the family \( \mathcal{G}_{str} = \{0, 1, A, A, A, A\} \) is an intuitionistic fuzzy \( \mathcal{G} \) structure on \( X \). Therefore, \((X, \mathcal{G}_{str})\) is an intuitionistic fuzzy \( \mathcal{G} \) structure space. Now,

(i) \( B = \{x, (a/0.7, b/0.7), (a/0.3, b/0.3)\} \) is an intuitionistic fuzzy \( semi_{\mathcal{G}_{str}} \) open set but need not be an intuitionistic fuzzy \( \alpha_{\mathcal{G}_{str}} \) open set in \( X \).

(ii) \( C = \{x, (a/0.2, b/0.3), (a/0.4, b/0.6)\} \) is an intuitionistic fuzzy \( \beta_{\mathcal{G}_{str}} \) open set but need not be an intuitionistic fuzzy \( pre_{\mathcal{G}_{str}} \) open set in \( X \).

Remark 4.6. From the following diagram, the following implications holds:

5. Separation axioms in an intuitionistic fuzzy \( \mathcal{G} \) structure space

Definition 5.1. Let \((X, \mathcal{G}_{str})\) be an intuitionistic fuzzy \( \mathcal{G} \) structure space. Then an intuitionistic fuzzy set \( A \) is said to be an intuitionistic fuzzy \( \mathcal{G}_{str} \) (resp. \( \alpha_{\mathcal{G}_{str}} \)) clopen set if and only if it is both intuitionistic fuzzy \( \mathcal{G}_{str} \) (resp. \( \alpha_{\mathcal{G}_{str}} \)) open and intuitionistic fuzzy \( \mathcal{G}_{str} \) (resp. \( \alpha_{\mathcal{G}_{str}} \)) closed.

Definition 5.2. Let \((X, \mathcal{G}_{str})\) be an intuitionistic fuzzy \( \mathcal{G} \) structure space. Then an intuitionistic fuzzy set \( A \) is said to be an intuitionistic fuzzy \( \alpha_{\mathcal{G}_{str}} \) exterior of \( A \) if

\[
IFExt_{\alpha_{\mathcal{G}_{str}}} (A) = IF_{\alpha_{\mathcal{G}_{str}}} int \left( \overline{A} \right).
\]

Remark 5.1. Let \((X, \mathcal{G}_{str})\) be an intuitionistic fuzzy \( \mathcal{G} \) structure space. Then

(i) If \( A \) is an intuitionistic fuzzy \( \alpha_{\mathcal{G}_{str}} \) clopen set, then \( IFExt_{\alpha_{\mathcal{G}_{str}}} (A) = A = \overline{A} \).

(ii) If \( A \subseteq B \) then \( IFExt_{\alpha_{\mathcal{G}_{str}}} (A) \supseteq IFExt_{\alpha_{\mathcal{G}_{str}}} (B) \)

(iii) \( IFExt_{\alpha_{\mathcal{G}_{str}}} (1) = 0 \) and \( IFExt_{\alpha_{\mathcal{G}_{str}}} (0) = 1 \).

Definition 5.3. Let \((X, \mathcal{G}_{1_{str}})\) and \((Y, \mathcal{G}_{2_{str}})\) be any two intuitionistic fuzzy \( \mathcal{G} \) structure space. Let \( f : (X, \mathcal{G}_{1_{str}}) \rightarrow (Y, \mathcal{G}_{2_{str}}) \) be an intuitionistic fuzzy function. Then
(i) intuitionistic fuzzy \( \alpha_{\mathcal{H}^{\mathcal{G}}} \) continuous function if for each intuitionistic fuzzy point \( x_{r,s} \) in \( X \) and each intuitionistic fuzzy \( \mathcal{G}_{2,\mathcal{G}} \) open set \( B \) in \( Y \) containing \( f(x_{r,s}) \), there exists an intuitionistic fuzzy \( \alpha_{\mathcal{H}^{\mathcal{G}}} \) open set \( A \) in \( X \) containing \( x_{r,s} \) such that \( f(A) \subseteq B \).

(ii) intuitionistic fuzzy \( \mathcal{E}_{\alpha_{\mathcal{H}^{\mathcal{G}}}} \) continuous function if for each intuitionistic fuzzy point \( x_{r,s} \) in \( X \) and each intuitionistic fuzzy \( \mathcal{G}_{2,\mathcal{G}} \) clopen set \( B \) in \( Y \) containing \( f(x_{r,s}) \), there exists an intuitionistic fuzzy \( \alpha_{\mathcal{H}^{\mathcal{G}}} \) clopen set \( A \) in \( X \) containing \( x_{r,s} \) such that

\[
\left( I F_{\mathcal{E}_{\alpha_{\mathcal{H}^{\mathcal{G}}}}} (A) \right) \subseteq B.
\]

(iii) intuitionistic fuzzy \( \alpha_{\mathcal{H}^{\mathcal{G}}} \) open function if for each intuitionistic fuzzy \( \mathcal{G}_{1,\mathcal{G}} \) open set \( A \) in \( X \), \( f^{-1}(A) \) is an intuitionistic fuzzy \( \alpha_{\mathcal{H}^{\mathcal{G}}} \) open set \( A \) in \( Y \).

**Proposition 5.1.** Let \( (X, \mathcal{G}_{1,\mathcal{G}}) \) and \( (Y, \mathcal{G}_{2,\mathcal{G}}) \) be any two intuitionistic fuzzy \( \mathcal{G} \) structure spaces. Let \( f : (X, \mathcal{G}_{1,\mathcal{G}}) \to (Y, \mathcal{G}_{2,\mathcal{G}}) \) be an intuitionistic fuzzy function. Then the following are equivalent

(i) \( f \) is intuitionistic fuzzy \( \mathcal{E}_{\alpha_{\mathcal{H}^{\mathcal{G}}}} \) continuous function.

(ii) \( f^{-1}(A) \) is an intuitionistic fuzzy \( \alpha_{\mathcal{H}^{\mathcal{G}}} \) open set in \( X \), for each intuitionistic fuzzy \( \mathcal{G}_{2,\mathcal{G}} \) clopen set \( A \) in \( Y \).

(iii) \( f^{-1}(A) \) is an intuitionistic fuzzy \( \alpha_{\mathcal{H}^{\mathcal{G}}} \) closed set in \( X \), for each intuitionistic fuzzy \( \mathcal{G}_{2,\mathcal{G}} \) clopen set \( A \) in \( Y \).

(iv) \( f^{-1}(A) \) is an intuitionistic fuzzy \( \alpha_{\mathcal{H}^{\mathcal{G}}} \) clopen set in \( X \), for each intuitionistic fuzzy \( \mathcal{G}_{2,\mathcal{G}} \) clopen set \( A \) in \( Y \).

**Proof.** (i) \( \Rightarrow \) (ii) Let \( B \) be an intuitionistic fuzzy \( \mathcal{G}_{2,\mathcal{G}} \) clopen set in \( Y \) and let \( x_{r,s} \in f^{-1}(B) \). Since \( f(x_{r,s}) \in B \), by (i), there exists an intuitionistic fuzzy \( \alpha_{\mathcal{H}^{\mathcal{G}}} \) clopen set \( A \) in \( X \) containing \( x_{r,s} \) such that

\[
f\left( I F_{\alpha_{\mathcal{H}^{\mathcal{G}}}} (A) \right) = f\left( I F_{\alpha_{\mathcal{H}^{\mathcal{G}}}} \text{ int}(\bar{A}) \right) \subseteq B \text{, } I F_{\alpha_{\mathcal{H}^{\mathcal{G}}}} \text{ int}(\bar{A}) \subseteq f^{-1}(B).
\]

This implies that \( f^{-1}(B) = \bigcup_{x_{r,s} \in I F_{\alpha_{\mathcal{H}^{\mathcal{G}}}} \text{ int}(\bar{A})} I F_{\alpha_{\mathcal{H}^{\mathcal{G}}}} \text{ int}(\bar{A}) \). Thus \( f^{-1}(B) \) is an intuitionistic fuzzy \( \alpha_{\mathcal{H}^{\mathcal{G}}} \) open set in \( X \).

(ii) \( \Rightarrow \) (iii) Let \( B \) be an intuitionistic fuzzy \( \mathcal{G}_{2,\mathcal{G}} \) clopen set in \( Y \). Then \( B \) is an intuitionistic fuzzy \( \mathcal{G}_{2,\mathcal{G}} \) clopen set in \( Y \). Thus, \( f^{-1}(B) = f^{-1}(B) \). Since \( f^{-1}(B) \) is an intuitionistic fuzzy \( \alpha_{\mathcal{H}^{\mathcal{G}}} \) open set in \( X \), \( f^{-1}(B) \) is an intuitionistic fuzzy \( \alpha_{\mathcal{H}^{\mathcal{G}}} \) closed set in \( X \).

(iii) \( \Rightarrow \) (iv) The proof is easily.
Let $B$ be an intuitionistic fuzzy $\mathcal{G}_{str}$ clopen set in $Y$ containing $f(x_{r,s})$. By (iv), $f^{-1}(B)$ is an intuitionistic fuzzy $\mathcal{G}_{str}$ clopen set in $X$. If we take $A = f^{-1}(B)$, then $f(A) \subseteq B$. By Remark 5.1, $f\left(\text{IFExt}_{\alpha_{d_{str}}}(A)\right) \subseteq B$.

**Remark 5.2.** Let $(X, \mathcal{G}_{str})$ and $(Y, \mathcal{G}_{str})$ be any two intuitionistic fuzzy $\mathcal{G}$ structure spaces. If $f : (X, \mathcal{G}_{str}) \rightarrow (Y, \mathcal{G}_{str})$ is an intuitionistic fuzzy $\alpha_{d_{str}}$ exterior set connected function, then $f$ is an intuitionistic fuzzy $\alpha_{d_{str}}$ continuous function.

**Definition 5.4.** An intuitionistic fuzzy $\mathcal{G}$ structure space $(X, \mathcal{G}_{str})$ is said to be an

(i) intuitionistic fuzzy $\mathcal{G}_{str}$ clo $T_0$ if for each pair of distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in $X$, there exists an intuitionistic fuzzy $\mathcal{G}_{str}$ clopen set $A$ of $X$ containing one intuitionistic fuzzy point $x_{r,s}$ but not $y_{m,n}$.

(ii) intuitionistic fuzzy $\mathcal{G}_{str}$ clo $T_1$ if for each pair of distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in $X$, there exist an intuitionistic fuzzy $\mathcal{G}_{str}$ clopen sets $A$ and $B$ containing $x_{r,s}$ and $y_{m,n}$ respectively such that $y_{m,n} \notin A$ and $x_{r,s} \notin B$.

(iii) intuitionistic fuzzy $\mathcal{G}_{str}$ clo $T_2$ if for each pair of distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in $X$, there exist an intuitionistic fuzzy $\mathcal{G}_{str}$ clopen sets $A$ and $B$ containing $x_{r,s}$ and $y_{m,n}$ respectively such that $A \cap B = \emptyset$.

(iv) intuitionistic fuzzy $\alpha_{d_{str}}$ clo -regular if for each intuitionistic fuzzy $\mathcal{G}_{str}$ clopen set $A$ and an intuitionistic fuzzy point $x_{r,s} \notin A$, there exist disjoint intuitionistic fuzzy $\alpha_{d_{str}}$ open sets $B$ and $C$ such that $A \subseteq B$ and $x_{r,s} \in C$.

(v) intuitionistic fuzzy $\alpha_{d_{str}}$ clo -normal if for each pair of disjoint intuitionistic fuzzy $\mathcal{G}_{str}$ clopen sets $A$ and $B$ in $X$, there exist disjoint intuitionistic fuzzy $\alpha_{d_{str}}$ open sets $C$ and $D$ such that $A \subseteq C$ and $B \subseteq D$.

**Definition 5.5.** An intuitionistic fuzzy $\mathcal{G}$ structure space $(X, \mathcal{G}_{str})$ is said to be an

(i) intuitionistic fuzzy $\alpha_{d_{str}}$ - $T_0$ if for each pair of distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in $X$, there exists an intuitionistic fuzzy $\alpha_{d_{str}}$ open set $A$ of $X$ containing one intuitionistic fuzzy point $x_{r,s}$ but not $y_{m,n}$.

(ii) intuitionistic fuzzy $\alpha_{d_{str}}$ - $T_1$ if for each pair of distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in $X$, there exist an intuitionistic fuzzy $\alpha_{d_{str}}$ open sets $A$ and $B$ containing $x_{r,s}$ and $y_{m,n}$ respectively such that $y_{m,n} \notin A$ and $x_{r,s} \notin B$.

(iii) intuitionistic fuzzy $\alpha_{d_{str}}$ - $T_2$ if for each pair of distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in $X$, there exist an intuitionistic fuzzy $\alpha_{d_{str}}$ open sets $A$ and $B$ containing $x_{r,s}$ and $y_{m,n}$ respectively such that $A \cap B = \emptyset$. 

(iv) intuitionistic fuzzy strongly $\alpha_{g\omega}$-regular if for each intuitionistic fuzzy $\alpha_{g\omega}$ closed set $A$ and an intuitionistic fuzzy point $x_{r,s} \not\in A$, there exist disjoint intuitionistic fuzzy $\mathcal{G}_{\omega}$ open sets $B$ and $C$ such that $A \subseteq B$ and $x_{r,s} \in C$.

(v) intuitionistic fuzzy strongly $\alpha_{g\omega}$-normal if for each pair of disjoint intuitionistic fuzzy $\alpha_{g\omega}$ closed sets $A$ and $B$ in $X$, there exist disjoint intuitionistic fuzzy $\mathcal{G}_{\omega}$ open sets $C$ and $D$ such that $A \subseteq C$ and $B \subseteq D$.

**Proposition 5.2.** Let $(X, \mathcal{G}_{1\omega})$ and $(Y, \mathcal{G}_{2\omega})$ be any two intuitionistic fuzzy $\mathcal{G}$ structure spaces. Let $f : (X, \mathcal{G}_{1\omega}) \rightarrow (Y, \mathcal{G}_{2\omega})$ be an injective, intuitionistic fuzzy $E_{\alpha_{g\omega}}$ continuous function.

(i) If $(Y, \mathcal{G}_{2\omega})$ is intuitionistic fuzzy $\mathcal{G}_{2\omega}$ clo-$T_0$, then $(X, \mathcal{G}_{1\omega})$ is intuitionistic fuzzy $\alpha_{g\omega}$ clo-$T_0$.

(ii) If $(Y, \mathcal{G}_{2\omega})$ is intuitionistic fuzzy $\mathcal{G}_{2\omega}$ clo-$T_1$, then $(X, \mathcal{G}_{1\omega})$ is intuitionistic fuzzy $\alpha_{g\omega}$ clo-$T_1$.

(iii) If $(Y, \mathcal{G}_{2\omega})$ is intuitionistic fuzzy $\mathcal{G}_{2\omega}$ clo-$T_2$, then $(X, \mathcal{G}_{1\omega})$ is intuitionistic fuzzy $\alpha_{g\omega}$ clo-$T_2$.

**Proof.** (i) Suppose that $(Y, \mathcal{G}_{2\omega})$ is an intuitionistic fuzzy $\mathcal{G}_{2\omega}$ clo-$T_0$ space. For any distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in $X$, there exists an intuitionistic fuzzy $\mathcal{G}_{2\omega}$ clopen set $A$ in $Y$ such that $f(x_{r,s}) \in A$ and $f(y_{m,n}) \notin A$. Since $f$ is an intuitionistic fuzzy $E_{\alpha_{g\omega}}$ continuous function, $f^{-1}(A)$ is an intuitionistic fuzzy $\alpha_{g\omega}$ open set in $X$ such that $x_{r,s} \in f^{-1}(A)$. This implies that $(X, \mathcal{G}_{1\omega})$ is an intuitionistic fuzzy $\alpha_{g\omega}$ clo-$T_0$ space.

(ii) Suppose that $(Y, \mathcal{G}_{2\omega})$ is an intuitionistic fuzzy $\mathcal{G}_{2\omega}$ clo-$T_1$ space. For any distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in $X$, there exists an intuitionistic fuzzy $\mathcal{G}_{2\omega}$ clopen sets $A$ and $B$ in $Y$ such that $f(x_{r,s}) \in A$, $f(x_{r,s}) \notin B$, $f(y_{m,n}) \in A$ and $f(y_{m,n}) \notin B$. Since $f$ is an intuitionistic fuzzy $E_{\alpha_{g\omega}}$ continuous function, $f^{-1}(A)$ and $f^{-1}(B)$ are intuitionistic fuzzy $\alpha_{g\omega}$ open sets in $X$ respectively such that $x_{r,s} \in f^{-1}(A)$, $x_{r,s} \notin f^{-1}(B)$, $y_{m,n} \notin f^{-1}(A)$ and $y_{m,n} \in f^{-1}(B)$. This implies that $(X, \mathcal{G}_{1\omega})$ is an intuitionistic fuzzy $\alpha_{g\omega}$ clo-$T_1$ space.

(iii) Suppose that $(Y, \mathcal{G}_{2\omega})$ is an intuitionistic fuzzy $\mathcal{G}_{2\omega}$ clo-$T_2$ space. For any distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in $X$, there exists an intuitionistic fuzzy $\mathcal{G}_{2\omega}$ clopen sets $A$ and $B$ in $Y$ such that $f(x_{r,s}) \in A$, $f(x_{r,s}) \notin B$, $f(y_{m,n}) \in A$, $f(y_{m,n}) \notin B$ and $A \cap B = \emptyset$. Since $f$ is an intuitionistic fuzzy $E_{\alpha_{g\omega}}$ continuous function, $f^{-1}(A)$
and 

\[ f^{-1}(B) \]

are intuitionistic fuzzy \( \alpha_{g_{tsr}} \) open sets in \( X \) containing \( x_{r,s} \) and \( y_{r,s} \) respectively such that

\[ f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(0_{\cdot}) = 0 \]

This implies that \( (X, g_{1str}) \) is an intuitionistic fuzzy \( \alpha_{g_{tsr}} \)-\( T_2 \) space.

**Proposition 5.3.** Let \( (X, g_{1str}) \) and \( (Y, g_{2str}) \) be any two intuitionistic fuzzy \( g \) structure spaces. Let \( f: (X, g_{1str}) \rightarrow (Y, g_{2str}) \) be an intuitionistic fuzzy function.

(i) If \( f \) is an injective, intuitionistic fuzzy \( \alpha_{q_{dr}} \) open function and intuitionistic fuzzy \( E_{\alpha q_{dr}} \) continuous function from intuitionistic fuzzy strongly \( \alpha_{g_{tsr}} \)-regular space \( (X, g_{1str}) \) onto an intuitionistic fuzzy \( g \) structure space \( (Y, g_{2str}) \), then \( (Y, g_{2str}) \) is an intuitionistic fuzzy \( \alpha_{g_{nts}} \) clo-regular space.

(ii) If \( f \) is an injective, intuitionistic fuzzy \( \alpha_{q_{dr}} \) open function and intuitionistic fuzzy \( E_{\alpha q_{dr}} \) continuous function from intuitionistic fuzzy strongly \( \alpha_{g_{tsr}} \)-normal space \( (X, g_{1str}) \) onto an intuitionistic fuzzy \( g \) structure space \( (Y, g_{2str}) \), then \( (Y, g_{2str}) \) is an intuitionistic fuzzy \( \alpha_{g_{nts}} \) clo-normal space.

**Proof.** (i) Let \( A \) be an intuitionistic fuzzy \( g_{2str} \) clopen set in \( Y \) such that \( y_{m,n} \notin A \). Take \( y_{m,n} = f(x_{r,s}) \). Since \( f \) is an intuitionistic fuzzy \( E_{\alpha q_{dr}} \) continuous function, \( B = f^{-1}(A) \) is an intuitionistic fuzzy \( \alpha_{g_{tsr}} \) closed sets in \( X \) such that \( x_{r,s} \notin B \). Since \( (X, g_{1str}) \) is an intuitionistic fuzzy strongly \( \alpha_{g_{tsr}} \)-regular space, there exist disjoint intuitionistic fuzzy \( g_{1str} \) open sets \( C \) and \( D \) such that \( B \subseteq C \) and \( x_{r,s} \in D \). Since \( f \) is an intuitionistic fuzzy \( \alpha_{g_{tsr}} \) open function, we have \( A = f(B) \subseteq f(C) \) and \( y_{m,n} = f(x_{r,s}) \in f(D) \) such that \( f(C) \) and \( f(D) \) are disjoint intuitionistic fuzzy \( \alpha_{g_{nts}} \) open sets in \( Y \). This implies that \( (Y, g_{2str}) \) is an intuitionistic fuzzy \( \alpha_{g_{nts}} \) clo-regular space.

(ii) Let \( A_1 \) and \( A_2 \) be disjoint intuitionistic fuzzy \( g_{2str} \) clopen sets in \( Y \). Since \( f \) is an intuitionistic fuzzy \( E_{\alpha q_{dr}} \) continuous function, \( f^{-1}(A_1) \) and \( f^{-1}(A_2) \) are intuitionistic fuzzy \( \alpha_{g_{tsr}} \) closed sets in \( X \). Take \( B = f^{-1}(A_1) \) and \( C = f^{-1}(A_2) \). This implies that \( B \cap C = 0_{\cdot} \). Since \( (X, g_{1str}) \) is an intuitionistic fuzzy strongly \( \alpha_{g_{tsr}} \)-normal space, there exist disjoint intuitionistic fuzzy \( g_{1str} \) open sets \( D_1 \) and \( D_2 \) such that \( B \subseteq D_1 \) and \( C \subseteq D_2 \). Since \( f \) is an intuitionistic fuzzy \( \alpha_{g_{tsr}} \) open function, we have \( A_1 = f(B) \subseteq f(D_1) \) and \( A_2 = f(C) \subseteq f(D_2) \) such that \( f(D_1) \) and \( f(D_2) \) are disjoint intuitionistic fuzzy \( \alpha_{g_{nts}} \) open sets in \( Y \). This implies that \( (Y, g_{2str}) \) is an intuitionistic fuzzy \( \alpha_{g_{nts}} \) clo-normal space.

**Definition 5.6.** Let \( (X, g_{1str}) \) and \( (Y, g_{2str}) \) be any two intuitionistic fuzzy \( g \) structure spaces. Let \( f: X \rightarrow Y \) be an intuitionistic fuzzy function. An intuitionistic
fuzzy graph $G(f) = \{(x_{r,s}, f(x_{r,s}) : x_{r,s} \in \xi^X \subseteq \xi^X \times \xi^Y \}$ of an intuitionistic fuzzy function $f$ is an intuitionistic fuzzy $\mathcal{G}_{\text{clo}}$-co-closed graph if and only if for each $(x_{r,s}, y_{m,n}) \in \xi^X \times \xi^Y \setminus G(f)$, there exist an intuitionistic fuzzy $\mathcal{G}_{\text{clo}}$ open set $A$ in $X$ containing $x_{r,s}$ and an intuitionistic fuzzy $\mathcal{G}_{2\text{str}}$ clopen set $B$ in $Y$ containing $y_{m,n}$ such that $f(A) \cap B = \emptyset$.

**Proposition 5.4.** Let $(X, \mathcal{G}_{1\text{str}})$, $(Y, \mathcal{G}_{2\text{str}})$ and $(X \times Y, \mathcal{G}_{2\text{str}})$ be any three intuitionistic fuzzy structure spaces. Let $f : (X, \mathcal{G}_{1\text{str}}) \to (Y, \mathcal{G}_{2\text{str}})$ be an intuitionistic fuzzy function.

(i) If $f$ is an intuitionistic fuzzy $E_{\mathcal{G}_{1\text{str}}}$ continuous function and $(Y, \mathcal{G}_{2\text{str}})$ is an intuitionistic fuzzy intuitionistic fuzzy $\mathcal{G}_{2\text{str}}$, $T_2$ space, then $G(f)$ is an intuitionistic fuzzy $\mathcal{G}_{\text{clo}}$-co-closed graph in $\xi^X \times \xi^Y$.

(ii) If $f$ is an intuitionistic fuzzy $\mathcal{G}_{\text{clo}}$ continuous function and $(Y, \mathcal{G}_{2\text{str}})$ is an intuitionistic fuzzy intuitionistic fuzzy $\mathcal{G}_{2\text{str}}$, $T_1$ space, then $G(f)$ is an intuitionistic fuzzy $\mathcal{G}_{\text{clo}}$-co-closed graph in $\xi^X \times \xi^Y$.

(iii) If $f$ is an intuitionistic fuzzy injective function has an intuitionistic fuzzy $\mathcal{G}_{\text{clo}}$-co-closed graph in $\xi^X \times \xi^Y$. Then $(X, \mathcal{G}_{1\text{str}})$ is an intuitionistic fuzzy $\mathcal{G}_{\text{clo}}$, $T_1$ space.

**Proof.** (i) Let $(x_{r,s}, y_{m,n}) \in \xi^X \times \xi^Y \setminus G(f)$ and $f(x_{r,s}) \notin y_{m,n}$. Since $(Y, \mathcal{G}_{2\text{str}})$ is an intuitionistic fuzzy $\mathcal{G}_{2\text{str}}$, $T_2$ space, there exist an intuitionistic fuzzy $\mathcal{G}_{2\text{str}}$ clopen sets $A$ and $B$ in $Y$ containing $f(x_{r,s})$ and $y_{m,n}$ respectively such that $A \cap B = \emptyset$. Since $f$ is an intuitionistic fuzzy $E_{\mathcal{G}_{1\text{str}}}$ continuous function, there exists an intuitionistic fuzzy $\mathcal{G}_{\text{clo}}$ open set $C$ in $X$ containing $x_{r,s}$ such that $f(\text{Ext}_{\mathcal{G}_{1\text{str}}}(C)) = f(C) \subseteq A$ and $f(C) \cap A = \emptyset$. This implies that $G(f)$ is an intuitionistic fuzzy $\mathcal{G}_{\text{clo}}$-co-closed graph in $\xi^X \times \xi^Y$.

(ii) Let $(x_{r,s}, y_{m,n}) \in \xi^X \times \xi^Y \setminus G(f)$ and $f(x_{r,s}) \notin y_{m,n}$. Since $(Y, \mathcal{G}_{2\text{str}})$ is an intuitionistic fuzzy $\mathcal{G}_{2\text{str}}$, $T_2$ space, there exist an intuitionistic fuzzy $\mathcal{G}_{2\text{str}}$ clopen set $A$ in $Y$ such that $f(x_{r,s}) \in A$ and $f(x_{r,s}) \notin A$. Since $f$ is an intuitionistic fuzzy $\mathcal{G}_{\text{clo}}$ continuous function, there exists an intuitionistic fuzzy $\mathcal{G}_{\text{clo}}$ open set $B$ in $X$ containing $x_{r,s}$ such that $f(B) \subseteq A$. Therefore, $f(B) \cap A = \emptyset$, and $A$ is an intuitionistic fuzzy $\mathcal{G}_{2\text{str}}$ clopen in $Y$ containing $y_{m,n}$. This implies that $G(f)$ is an intuitionistic fuzzy $\mathcal{G}_{\text{clo}}$-co-closed set in $\xi^X \times \xi^Y$.

(iii) Let \( x_{r,s} \) and \( y_{m,n} \) be any two intuitionistic fuzzy points \( X \). Then \( (x_{r,s}, f(x_{r,s})) \in \xi^X \times \xi^Y \setminus G(f) \). Since \( G(f) \) is an intuitionistic fuzzy \( \alpha_{\Theta_{2str}} \)-co-closed graph in \( \xi^X \times \xi^Y \), there exist an intuitionistic fuzzy \( \alpha_{\Theta_{2str}} \)-open set \( A \) in \( X \) and an intuitionistic fuzzy \( \Theta_{2str} \)-clopen set \( B \) in \( Y \) such that \( x_{r,s} \in A \), \( f(y_{m,n}) \in B \) and \( f(A) \cap B = 0 \). Since \( f \) is an intuitionistic fuzzy injective function, \( A \cap f^{-1}(B) = 0 \). \( f^{-1}(B) \) is an intuitionistic fuzzy \( \Theta_{2str} \)-clopen set in \( Y \) containing \( y_{r,s} \). And \( y_{m,n} \notin A \). Thus \( (X, \Theta_{1str}) \) is an intuitionistic fuzzy \( \alpha_{\Theta_{1str}} - T_1 \) space.

References

A Note on Fuzzy Continuum on Mixed Fuzzy Topological Spaces

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Abstract:
This paper deals with the study of \( H \)-continuum and \( N \)-continuum in Mixed Fuzzy Topological Spaces. We have already given the \( H \)-continuum and \( N \)-continuum structure to Fuzzy topological spaces in our papers “A note on \( H \)-continuum” and “Some Aspects of Fuzzy \( N \)-continuum”, and Prof. N. R. Das and P. C. Baishya has given the Mixed Topological structure in Fuzzy setting. We have tried to study the continuum structure in Mixed Fuzzy Topology. Hence in this paper we also see that the presence of \( H \)-closed and \( N \)-closed sets with Mixed Fuzzy topology give arise some new properties which are also be studied. Our perspective is to explore more results in this direction.

Keywords:
\( \theta \)-connectedness, \( \delta \)-connectedness, \( H \)-closed, \( N \)-closed, \( H \)-continuum, \( N \)-continuum, \( \theta \)-continuous, \( \delta \)-continuous.

1. Introduction

The study of mixed topology originated from the work of Alexiewicz and Semadini, N. R. Das and P. C. Baishya have constructed a fuzzy topology called Mixed Fuzzy Topology with the help of two given fuzzy topologies on a set \( X \). S. Ganguly and S. Jana introduced \( H \)-continuum and \( N \)-continuum in general topological spaces. We in our paper have given these two structure to fuzzy topological spaces. \( H \)-continuum is a \( \theta \)-connected \( H \)-set and \( N \)-continuum is a \( \delta \)-connected \( N \)-closed set.

In this paper we have combined our continuum structure with the mixed fuzzy topological space. If a fuzzy set is a \( H \)-set under one of the topology then it becomes
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$H$-set under the mixed topology also. Under some conditions if it an $H$-set under the other topology then also it an $H$-set under the mixed topology. These results will justify that our study to explore more results in this direction is fruitful.

2. Preliminaries

Most of the concepts, notations and definitions which we have used in this paper are standard by now. But for the sake of completeness, we recall some definitions and results used in the sequel. The remaining definitions and notations which are not explained can be referred to [6,7,8].

Definition 2.1. A fuzzy set in $X$ is called a fuzzy point denoted by $x_\lambda$ if it takes value $\lambda$ at $x$ and 0 at else.

Definition 2.2. A fuzzy point $x_\lambda$ is said to be $q$-coincident with a fuzzy set $A$ in $X$ if $\lambda + A(x) > 1$.

Definition 2.3. A fuzzy set $A$ is said to be quasi-coincident with another fuzzy set $B$ in $X$ if there exists $x \in X$ such that $A(x) + B(x) > 1$.

Definition 2.4. A fuzzy set $A$ in $(X, \tau)$ is called a $q$-nbd of $x_\lambda$ if and only if there exists a $B \in \tau$ such that $x_\lambda q B \leq A$.

Definition 2.5. A fuzzy point $x_\lambda$ is said to be a fuzzy $\theta$-cluster point of a fuzzy set $A$ in $(X, \tau)$ if the closure of every $q$-nbd of $x_\lambda$ is quasi-coincident with $A$.

The union of all $\theta$-cluster points of $A$ is called the $\theta$-closure of $A$ and is denoted by $A^\theta$.

Definition 2.6. A fuzzy point $x_\lambda$ is said to be a fuzzy $\delta$-cluster point of a fuzzy set $A$ in $(X, \tau)$ if every fuzzy regular open $q$-nbd $U$ of $x_\lambda$ is quasi-coincident with $A$.

The collection of $\delta$-cluster point if $A$ is called $\delta$-closure of $A$ and is denoted by $A^\delta$.

Definition 2.7. A triplet $(X, \tau_1, \tau_2)$ of a non-empty set $X$ together with tow fuzzy topologies $\tau_1$, $\tau_2$ is called a fuzzy bitopological space.

Definition 2.8. Let $(X, \tau_1)$ and $(X, \tau_2)$ be two fuzzy topological sapces and $\tau_1(\tau_2) = \{ A \in I^X | \text{for each } x_\lambda q A, \text{there exists a } \tau_1 q \text{-nbd } A_\lambda \text{ of } x_\lambda \text{ such that } \tau_1 \text{-closure of } A_\lambda \leq A \}$ . Then $\tau_1(\tau_2)$ is a topology on $X$ called a mixed fuzzy topology.
**Theorem 2.9** [1]. If \((x, \tau_1)\) and \((X, \tau_2)\) be two fuzzy topological spaces, then the mixed fuzzy topology \(\tau_1(\tau_2)\) is coarser than \(\tau_2\) i.e. \(\tau_1(\tau_2) \subseteq \tau_2\).

**Theorem 2.10** [1]. If \((x, \tau_1)\) and \((X, \tau_2)\) be two fuzzy topological spaces, with \(\tau_1\) fuzzy regular and \(\tau_1 \subset \tau_2\), then \(\tau_1 \subseteq \tau_1(\tau_2)\).

**Definition 2.11.** A pair of fuzzy sets \((P, Q)\) are said to be \(\theta\)-separation (\(\delta\)-separation) of \(A\) with respect to \(X\), if \(P \cup Q = A\) and \(P^\theta \cap Q = P \cap Q^\delta = 0\) \(\left(P^\delta \cap Q = P \cap Q^\delta = 0\right)\).

Notation. Let \(A\) be a fuzzy subset in a fuzzy bitopological space \((X, \tau_1, \tau_2)\). Then the \(\theta\)-closure of \(A\) with respect to \(\tau_1\) (respectively \(\delta\)-closure of \(A\) with respect to \(\tau_1\)) is denoted by \(\overline{A}^{\theta(\tau_1)}\) (resp. \(\overline{A}^{\delta(\tau_1)}\)). If there is only one topology we simply denoted it by \(\overline{A}^\theta\) (resp. \(\overline{A}^\delta\)).

3. **H-continuum in mixed fuzzy topological spaces**

**Theorem 3.1.** Let \((X, \tau_1)\) and \((X, \tau_2)\) be two fuzzy topological spaces and \(A\) be a fuzzy subset of \(X\). Then

\[
\overline{A}^{\theta(\tau_2)} \subseteq \overline{A}^{\theta(\tau_1)}
\]

That is the \(\theta\)-closure of \(A\) w.r.t. \(\tau_2\) is a fuzzy subset of the \(\theta\)-closure of \(A\) w.r.t. \(\tau_1(\tau_2)\).

**Proof.** Let \(x_\alpha \in \overline{A}^{\theta(\tau_1)}\). Then \(x_\alpha\) is a \(\tau_2\)-\(\theta\)-cluster point of \(A\) i.e. \(\tau_2\)-closure of every \(q\)-nbd \(N\) of \(x_\alpha\) is \(q\)-coincident with \(A\). Now if \(N_{x_\alpha}\) is a \(\tau_1(\tau_2)\) \(q\)-nbd of \(x_\alpha\) then there exists a fuzzy set \(V \in \tau_1(\tau_2)\) such that \(x_\alpha \in V \subseteq N_{x_\alpha}\). Since \(\tau_1(\tau_2) \subseteq \tau_2\) [Theorem 2.9], therefore \(V \in \tau_2\). Thus \(N_{x_\alpha}\) is a \(q\)-nbd of \(x_\alpha\) in \(\tau_2\) also. Since, \(x_\alpha\) is a \(\tau_2\)-\(\theta\)-cluster point of \(A\), therefore \(\overline{N}_{x_\alpha} \subseteq \overline{A}^{\theta(\tau_1)}\). Hence \(\overline{N}_{x_\alpha} \subseteq \overline{A}^{\theta(\tau_1)}\). Again \(\tau_1(\tau_2) \subseteq \tau_2\) implies \(\overline{N}^{\tau_2(\tau_1)} \subseteq \overline{A}^{\theta(\tau_1)}\). Therefore \(\overline{N}^{\tau_2(\tau_1)}(x)+A(x) > 1\), that is \(\tau_1(\tau_2)\)-closure of \(N_{x_\alpha}\) is \(q\)-coincident with \(A\), implies that \(x_\alpha \in \overline{A}^{\theta(\tau_1)}\). Hence the proof.

**Theorem 3.2.** If \((X, \tau_1)\) and \((X, \tau_2)\) be two fts and \(A\) be a fuzzy subset of \(X\) then

\[
\overline{A}^{\delta(\tau_2)} \subseteq \overline{A}^{\delta(\tau_1)}
\]

**Proof.** Similar with the proof of Theorem 3.1.
Theorem 3.3. Let \((X, \tau_1)\) and \((X, \tau_2)\) be two fts such that \(\tau_1\) is regular and \(\tau_1 \subset \tau_2\). Then \(\overline{A}^{\theta(\tau_2)} \leq \overline{A}^{\theta(\tau_1)}\) and \(\overline{A}^{\delta(\tau_2)} \leq \overline{A}^{\delta(\tau_1)}\).

Proof. Follows from the fact that \((X, \tau_1)\) by Theorem 2.10.

Theorem 3.4. If \((P, Q)\) is a \(\theta\)-separation (resp. \(\delta\)-separation) in \(\tau_1(\tau_2)\), then \((P, Q)\) is a \(\theta\)-separation (resp. \(\delta\)-separation) in \(\tau_2\).

Proof. If possible, suppose \((P, Q)\) is not a \(\theta\)-separation in \(\tau_2\). Then \(\overline{P}^{\theta(\tau_2)} \cap Q \neq 0\) or \(P \cap \overline{Q}^{\theta(\tau_2)} \neq 0\). Without any loss of generality, suppose \(\overline{P}^{\theta(\tau_2)} \cap Q \neq 0\). Since \(\overline{P}^{\theta(\tau_2)} \leq \overline{P}^{\theta(\tau_1)}\), therefore, \(\overline{P}^{\theta(\tau_1)} \cap Q \neq 0\). It is a contradiction to the fact that \((P, Q)\) is \(\theta\)-connected in \(\tau_1(\tau_2)\). Hence the result.

Theorem 3.5. If \(A\) is an \(H\)-subcontinuum (resp. \(N\)-subcontinuum) in \(\tau_2\), then \(A\) is \(H\)-subcontinuum (resp. \(N\)-subcontinuum) in \(\tau_1(\tau_2)\).

Proof. Let \(A\) be an \(H\)-subcontinuum in \((X, \tau_2)\). Then \(A\) is an \(H\)-set in \((X, \tau_2)\). Let \(\{U_\alpha : \alpha \in \Lambda\}\) be an open cover for \(A\) in \(\tau_1(\tau_2)\). Since \(\tau_1(\tau_2) \subset \tau_2\), therefore \(\{U_\alpha : \alpha \in \Lambda\}\) is a \(\tau_2\)-open cover for \(A\) also. As \(A\) is an \(H\)-set in \(\tau_2\) so there exists a subfamily \(\{U_\alpha : i = 1, 2, \ldots, n\}\) such that \(A \leq \bigvee_{i=1}^n U_{\alpha_i}\). But \(U_{\alpha_i} \leq U_{\alpha_i}^{\theta(\tau_2)}\), for each \(\alpha_i\). This implies that \(A\) is an \(H\)-set in \(\tau_1(\tau_2)\). Again from the Theorem 3.4 we have that \(A\) is \(\theta\)-connected in \(\tau_2\). Thus \(A\) is \(\theta\)-connected in \(\tau_1(\tau_2)\). Hence \(A\) is an \(H\)-subcontinuum in \((X, \tau_1(\tau_2))\).

Theorem 3.6. If \(\tau_1\) is regular and \(\tau_1 \subset \tau_2\), then if \(A\) is an \(H\)-subcontinuum in \(\tau_1(\tau_2)\) then it is \(H\)-subcontinuum in \(\tau_1\).

Proof. Similar to the proof of Theorem 3.5, as in this case \(\tau_1 \subset \tau_1(\tau_2)\).

4. \(\theta\)-continuous

Definition 4.1. A function \(f : X \to Y\) from an fts \((X, \tau_1)\) to another fts \((Y, \tau_2)\) is called \(\theta\)-continuous if for each fuzzy point \(x_\lambda\) in \(X\) and for each open \(q\)-nbd \(V\) of \(f(x_\lambda)\), there exists an open \(q\)-nbd \(U\) of \(X_\lambda\) such that \(f(U) \leq V\).
A Note on Fuzzy Continuum on Mixed Fuzzy Topological Spaces

Theorem 4.2. Let \((X, \tau_1, \tau_2)\) and \((Y, \tau'_1, \tau'_2)\) be two fuzzy bitopological spaces. If \(f : X \rightarrow Y\) is \(\tau_1-\tau'_1\) and \(\tau_2-\tau'_2\) \(\theta\)-continuous then \(f\) is \(\tau_1(\tau_2)-\tau'_1(\tau'_2)\) \(\theta\)-continuous.

Proof. Let \(x\) be a fuzzy point in \(X\) and \(V\) be a \(\tau'_1(\tau'_2)\) open \(-\)nbd of \(f(x)\).
Since \(\tau'_1(\tau'_2) \subseteq \tau'_2\), therefore \(V\) is \(\tau'_2\)-open \(-\)nbd of \(f(x)\).
Again \(f\) is \(\tau_2-\tau'_2\) \(\theta\)-continuous then there exists a \(\tau_2\)-open \(-\)nbd \(U\) of \(x\) such that \(f(U) \subseteq V\).
Again since \(f\) is \(\tau_1-\tau'_1\) \(\theta\)-continuous, therefore for any \(\tau'_1\)-open \(-\)nbd \(V_i\) of \(f(x)\) there exists a \(\tau_1\)-open \(-\)nbd \(U_i\) of \(x\) such that \(f(U_i) \subseteq V_i\).

Now let \(U = f^{-1}(V)\) and \(x\) be quasi-coincident with \(U\). Then \(\lambda + U(x) > 1\) which implies \(\lambda + f(U)(x) > 1\). I.e. \(f(x)\) is quasi-coincident \(-\)V. Since \(V\) is \(\tau'_1(\tau'_2)\) open, so there exists a \(\tau'_2\)-q \(-\)nbd \(V'_2\) of \(f(x)\) such that \(V'_2 \subseteq V\).
\(f\) is \(\tau_1-\tau'_1\) \(-\)continuous implies that there exists a \(\tau_1\)-open \(-\)nbd \(U_1\) of \(x\) such that \(f(U_1) \subseteq V'_1 \subseteq V\) implies \(U_1 \subseteq U\).

Again for each fuzzy point \(x\) in \(X\) and each \(\tau'_2\)-q \(-\)nbd \(V'_a\) of \(f(x)\), there exists a \(\tau_2\)-q \(-\)nbd \(U_2\) of \(x\) such that \(f(U_2) \subseteq V'_2\) implies \(U_2 \subseteq f^{-1}(V'_2)\).
Since \(U_2\) is \(\tau_2\)-q \(-\)nbd of \(x\) therefore we have a \(\tau_2\)-open set \(A\) such that \(x_\alpha \subseteq U_2\).
Therefore \(A_\beta \subseteq U_2 \subseteq f^{-1}(V'_2)\). Hence \(U \in \tau_1(\tau_2)\).
Moreover \(U = f^{-1}(V)\) implies \(f(U) \subseteq V\) implies \(f(U) \subseteq V'_1 \subseteq V\) implies \(U_1 \subseteq U\).

Proposition 4.3. Let \((X, \tau_1, \tau_2)\) and \((Y, \tau'_1, \tau'_2)\) be two fuzzy bitopological spaces. If \(f : X \rightarrow Y\) is \(\tau_1-\tau'_1\) and \(\tau_2-\tau'_2\), \(\delta\)-continuous then \(f\) is \(\tau_1(\tau_2)-\tau'_1(\tau'_2)\) \(\delta\)-continuous.

Proof. Similar with the proof of Theorem 4.2.

Proposition 4.4. Let \((X, \tau_1, \tau_2)\) and \((Y, \tau'_1, \tau'_2)\) be two fuzzy bitopological spaces where \(\tau'_1 \subseteq \tau'_2\) and \(\tau'_2\) is fuzzy regular. If \(f : X \rightarrow Y\) is \(\tau_1(\tau_2)-\tau'_1(\tau'_2)\) \(\theta\)-continuous then \(f\) is \(\tau_2-\tau'_1\) \(\theta\)-continuous.

Proof. Let \(B'\) be an \(\tau'_1\)-open \(-\)nbd of \(f(x)\).
Since \(\tau'_1 \subseteq \tau'_2\) and \(\tau'_2\) is fuzzy regular, therefore by Theorem 2.10, \(\tau'_1 \subseteq \tau'_2(\tau'_2)\).
Therefore, \(B'\) is a \(\tau'_1(\tau'_2)\)-open \(-\)nbd of \(f(x)\).
Now \(f\) is \(\tau_1(\tau_2)-\tau'_1(\tau'_2)\) \(\theta\)-continuous implies that there exists a \(\tau_1(\tau_2)\)-open \(-\)nbd \(B\) of \(x\) such that \(f(B) \subseteq \overline{B}\).
Again, \(\tau_1(\tau_2) \subseteq \tau_2\).
implies that $B$ is $\tau_2$-open and $f(B^\circ) \subseteq f(B^{(\tau_2)}) \subseteq \overline{B^{(\tau_1)}} \subseteq \overline{B^\circ}$. Hence $f$ is $\tau_2^* - \tau_1^*$ $\theta$-continuous.

**Proposition 4.5.** Let $(X, \tau_1, \tau_2)$ and $(Y, \tau_1^*, \tau_2^*)$ be two fuzzy bitopological spaces. If $f : X \to Y$ is $\tau_1(\tau_2) - \tau_1^*(\tau_2^*)$ $\theta$-continuous then $f$ is $\tau_2 - \tau_1^*(\tau_2^*)$ $\theta$-continuous.

**Proof.** Let $B^*$ be a $\tau_1^*(\tau_2^*)$ open $\theta$-nbd of $f(x)$ in $Y$. Then there exists a $\tau_1(\tau_2)$ open $\theta$-nbd $B$ of $x$ such that $f(B^\circ) \subseteq f(B^{(\tau_1^*)}) \subseteq \overline{B^{(\tau_2^*)}}$. Now by Theorem 2.9, we have $\tau_1(\tau_2) \subseteq \tau_2$. So, $B$ is $\tau_2$-open of $x$ and $f(B^\circ) \subseteq f(B^{(\tau_1^*)}) \subseteq \overline{B^{(\tau_2^*)}}$. So, $f$ is $\tau_1^*(\tau_2^*)$ $\theta$-continuous.

**Proposition 4.6.** Let $(X, \tau_1, \tau_2)$ and $(Y, \tau_1^*, \tau_2^*)$ be two fuzzy bitopological spaces. If $f : X \to Y$ is $\tau_1 - \tau_1^*$ $\theta$-continuous, and $\tau_1$ is fuzzy regular with $\tau_1 \subseteq \tau_2$, then $f$ is $\tau_1(\tau_2) - \tau_1^*$ $\theta$-continuous.

**Proof.** Let $B^*$ be a $\tau_1^*$ open $\theta$-nbd of $f(x)$ in $Y$. Then there exists a $\tau_1$ open $\theta$-nbd $B$ of $x$ such that $f(B^\circ) \subseteq f(B^{(\tau_1^*)}) \subseteq \overline{B^{(\tau_2^*)}}$. Now by Theorem 2.10, we have $\tau_1 \subseteq \tau_1(\tau_2)$. So, $B$ is $\tau_1(\tau_2)$-open and $f(B^\circ) \subseteq f(B^{(\tau_1^*)}) \subseteq \overline{B^{(\tau_2^*)}}$. Hence, $f$ is $\tau_1(\tau_2) - \tau_1^*$ $\theta$-continuous.

**Proposition 4.7.** If $(X, \tau_1^*, \tau_2^*)$, $(Y, \tau_1^*, \tau_2^*)$ are fuzzy bitopological space and $f : X \to Y$ is $\tau_1(\tau_2) - \tau_1^*$ $\theta$-continuous, then $f$ is $\tau_2 - \tau_2^*$ $\theta$-continuous.

**Proof.** If $B^*$ is a $\tau_1(\tau_2)$ open $\theta$-nbd of $f(x)$ in $Y$, then there exists a $\tau_1$ open $\theta$-nbd $B$ of $x$ such that $f(B^\circ) \subseteq f(B^{(\tau_1^*)}) \subseteq \overline{B^{(\tau_2^*)}}$. Now, by Theorem 2.9, we have $\tau_1(\tau_2) \subseteq \tau_2$, which implies $B$ is $\tau_2$-open and $f(B^\circ) \subseteq f(B^{(\tau_1^*)}) \subseteq \overline{B^{(\tau_2^*)}}$. Hence the result.

**Proposition 4.8.** Let $(X, \tau_1, \tau_2)$ and $(Y, \tau_1^*, \tau_2^*)$ be two fuzzy bitopological spaces. If $f : X \to Y$ is $\tau_1(\tau_2) - \tau_1^*$ $\theta$-continuous, then $f$ is $\tau_2 - \tau_2^*$ $\theta$-continuous.

**Proof.** If $B^*$ be a $\tau_2^*$ open $\theta$-nbd of $f(x)$ in $Y$. Then there exists a $\tau_1(\tau_2)$-open $\theta$-nbd $B$ of $x$ in $X$ such that $f(B^\circ) \subseteq f(B^{(\tau_1^*)}) \subseteq \overline{B^{(\tau_2^*)}}$. But according to Theorem 2.9, $\tau_1(\tau_2) \subseteq \tau_2$. So, $B \in \tau_2$ and $f(B^\circ) \subseteq f(B^{(\tau_1^*)}) \subseteq \overline{B^{(\tau_2^*)}}$. This proves the result.
5. $\theta$-open mappings

**Definition 5.1.** A function $f: X \to Y$ where $(X, \tau_1)$ and $(X, \tau_2)$ are two fuzzy topological spaces, is called a $\theta$-open map if for each $\theta$-open set $A$ of $X$, $f(A)$ is $\theta$-open in $Y$.

**Theorem 5.2.** Let $(X, \tau)$ and $(Y, \tau')$ be two fuzzy topological spaces. Then if $f: X \to Y$ is an onto fuzzy open map, then it is $\theta$-open.

**Proof.** Let $A$ be a $\theta$-open subset in $X$ and $f(x) \in f(A)$. This implies $x_\theta \in A$ and $A$ is $\theta$-open implies there exists a $q$-nbh $V_x$ of $x_\theta$ such that

$$V \subseteq \overline{V_x} \subseteq A$$

(1)

$V_x$ is a $q$-nbh of $x_\theta$ implies there exists an open subset $U$ of $(X, \tau)$ such that $x_\theta \overline{q} U \subseteq V_x$. Now, $x_\theta \overline{q} U$ implies $f(x) \overline{q} f(U)$ and $f(U) \subseteq f(V_x)$. Since, $f$ is open such that $f(U)$ is a open fuzzy set in $Y$ and so $f(V_x)$ is a $q$-nbh of $f(x)$. Again, $\overline{f(V_x)} \subseteq f(\overline{V_x})$. Therefore, (1) implies $\overline{f(V_x)} \subseteq f(A)$. So, $f(A)$ is $\theta$-open in $Y$.

**References**

Fuzzy Reliability Optimization Based on Fuzzy Geometric Programming Method Using Different Operators

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Abstract:
In this paper, we summarize the fundamentals of fuzzy GP and present the problem of optimal reliability for a series system subject to a cost constraint. In real life, it is necessary to improve the reliability of the system under limited available cost of reliability component. Practically some desired system reliability level subject to constraint on cost has always been imprecise and vague in nature. It may be formulated as a fuzzy geometric programming problem. Numerical examples are given to illustrate the model through fuzzy geometric programming by max-min, max-additive and max-product operators.

Key words:
Fuzzy GP, reliability, posynomial, signomial.

1. Introduction

To achieve an aim to find out the best way to increase the systems reliability, reliability optimization provides immense help to the reliability engineer. The reliability optimization is an important research topic in engineering and operation research. In practical field, the problem of series system reliability may be formed as a typical non-linear programming problem with non-linear cost-functions in fuzzy environment. Some researchers applied the fuzzy set theory to reliability analysis. Park [12] used fuzzy in the reliability apportionment problem for a two-component series system subject to a single constraint and solved it by FNLP technique. GP method is rarely used to solve the reliability optimization problem. Federowiez and Mazumder [1] first used GP on reliability optimization problem. Govil [11] used GP for a 3-stage series reliability system. Govi [10] also applied GP method on an optimal maintainability problem of a series system with cost constraint. Mahapatra and Roy [8] discussed optimal fuzzy reliability for a series system with cost constraint using fuzzy geometric programming.
Again Mahapatra and Roy [7] used geometric programming to analyze system reliability. But fuzzy reliability optimization model through $FGP$ is very rare in literature.

The Reliability optimization assumes that systems have redundancy components in series parallel or parallel systems and that alternative designs achieve the goal of optimal system reliability by optimal allocation of redundancy components. Reliability of a multi-stage system can be improved by adding similar components as redundancy to each subsystem, may be some different components that can be considered as design alternatives in a subsystem. Thus the problem is to improve system’s reliability associated with a system design under the limited availability resources. Tillman et al. [5], Singh and Misra [9] and Kuo et al. [18] illustrated allocation of redundant component in a system to enhance the system reliability, which is important in reliability engineering. In many practical problems, most of the parameters of an optimization model are not known exactly. Due to this imperfect and unreliable input information, fuzzy numbers become an important aspect in the reliability design of the engineering systems. The problem is to find out the optimum number of redundancies of similar components, which maximize the system reliability subject to the available system cost and weight, regards as the problem of geometric programming in the circumstance of fuzzy number presentation as relativity, cost and weight of components.

**Geometric Programming**

Geometric programming ($GP$) can be considered to be an innovative modus operandi to solve a nonlinear problem in comparison with other nonlinear techniques. It was originally developed to design engineering problems. It has become a very popular technique since its inception in solving non-linear problems. The concept of geometric programming ($GP$) was introduced by Duffin et al. [15] in their famous book *Geometric Programming-Theory and Application*. This publication is a landmark in the development of $GP$. It studied all the theoretical developments up to date providing important examples to illustrate the technique. In addition to elegant proofs, it provided several constructive transformations and approximation for expressing optimization problems in suitable form in order to solve by $GP$.

Many real-world problems comprise of positive as well as negative coefficients for the cost terms. However the study of $GP$ by Duffin et al. [15] deals with the problem involving only a positive coefficient for the component cost terms. Passy and Wilde [17] made a significant methodological development of $GP$ to deal with this type of problem. They extended the concept of the $GP$ technique to generalized polynomials free from a restrictive environment. Now $GP$ is capable of dealing with any problems involving signomials in both objective and constraint functions. It is important to note that any nonlinear algebraic problem can be transformed into an equivalent posynomial/signomial. For a detailed discussion, one may consult with the book *Applied Geometric Programming* written by Beightle and Phillips [6].

The advantages of $GP$ are as follows:

- This technique provides us with a systematic approach for solving a class of nonlinear optimization problems by finding the optimal value of the objective function and then the optimal values of the design variable are derived.
- This method often reduces a complex nonlinear optimization problem to a set of simultaneous equations.
- This approach is more amenable to the digital computers.
This method allows an easy sensitivity analysis, which can be performed in the optimal solution.

\( GP \) inherits some drawbacks. The main disadvantages of \( GP \) lie in the fact that it requires the objective functions and constraints in the form of posynomials/signomials.

**Note.** Someone guesses that the name \( GP \) comes from the many geometrical problems. There is a difference between \( GP \) and geometric optimization (\( GOP \)). \( GP \) is an optimization based on the arithmetic-geometric mean inequality (\( A.M. \geq G.M. \)). However, \( GOP \) is an optimization problem involving geometry.

\( GP \) is an optimization problem of the form

\[
\text{Minimize } g_0(t) \tag{1.1}
\]

Subject to \( g_j(t) \leq 1, \ j = 1,2,\cdots, m \), \( h_k(t) = 1, \ k = 1,2,\cdots, p \), \( t_i > 0 \), \( i = 1,2,\cdots, n \)

where \( g_j(t) \ (j = 0,1,2,\cdots, m) \) are posynomial or signomial functions, \( h_k(t) \ (k = 1,2,\cdots, p) \) are monomials and \( t \) is the decision variable vector of \( n \) components \( t_i \ (i = 1,2,\cdots, n) \).

The problem (1.1) may be written as:

\[
\text{Minimize } g_0(t) \tag{1.2}
\]

Subject to \( g_j(t) \leq 1 \), \( i = 1,2,\cdots, m \), \( t > 0 \), [since \( g_j(t) \leq 1, \ h_j(t) = 1 \Rightarrow g_j(t) \leq 1 \) where \( g_j(t) = g_j(t)/h_j(t) \) be a posynomial \( j = 1,2,\cdots, m; k = 1,2,\cdots, p \)].

2. Posynomial geometric programming problem

2.1 Primal problem

\[
\text{Minimize } g_0(t) \tag{2.1}
\]

Subject to \( g_j(t) \leq 1 \ (j = 1,2,\cdots, m) \) and \( t > 0 \), \( (i = 1,2,\cdots, n) \) where \( g_j(t) = \sum_{k=1}^{N_j} c_{j,k} \prod_{i=1}^{n} t_i^{\alpha_{j,k,i}} \)

Where \( c_{j,k} > 0 \) and \( \alpha_{j,k,i} \ (i = 1,2,\cdots, n; k = 1,2,\cdots, N_j; j = 0,1,2,\cdots, m) \) are real numbers. \( t = (t_1,t_2,\cdots,t_n)^T \).

It is a constrained posynomial primal geometric problem (\( PGP \)). The number of inequality constraints in the problem (2.1) is \( m \). The number of terms in each posynomial constraint function varies, and it is denoted by \( N_j \) for each \( j = 0,1,2,\cdots, m \).

The degree of difficulty (\( DD \)) of a \( GP \) is defined as number of terms in a \( PGP \) - (number of variables in \( PGP + 1 \)).

2.2 Dual problem

The dual programming of (2.1) is as follows:
Maximize \( d(w) = \prod_{j=0}^{m} \prod_{k=1}^{N_j} \left( \frac{c_{jk} w_{jk}}{w_{jk}} \right)^{w_{jk}} \) \hspace{1cm} (2.2)

Subject to \( \sum_{k=1}^{N_j} w_{jk} = 1 \) (normality condition) \( \sum_{j=1}^{m} \sum_{k=1}^{N_j} \alpha_{j,k} w_{jk} = 0, \ (i = 1, 2, \cdots, n) \) (orthogonality condition) \( w_{j0} = \sum_{k=1}^{N_j} w_{jk} \geq 0, \ w_{jk} \geq 0, \ (i = 1, 2, \cdots, n; \ k = 1, 2, \cdots, N_j), \ w_{j0} = 1. \)

There are \( n+1 \) independent dual constraint equalities and \( N = \sum_{j=1}^{m} N_j \) independent dual variables for each term of the primal problem. In this case \( DD = N - (n+1) \).

3. Signomial geometric programming problem

3.1 Primal problem

\[ \text{Minimize} \ g_0(t) \] \hspace{1cm} (3.1)

Subject to \( g_j(t) \leq \delta_j, \ (j = 1, 2, \cdots, m) \) and \( t \geq 0, \ (i = 1, 2, \cdots, n) \) where \( g_j(t) = \sum_{k=1}^{N_j} \delta_{j,k} c_{jk} \prod_{j=1}^{m} t_{j,k} \delta_{j,k} (j = 0, 1, 2, \cdots, m), \ \delta_{j,k} = 1 \ (j = 2, \cdots, m), \ \delta_{j,k} = 1 \ (j = 0, 1, 2, \cdots, m; \ k = 1, 2, \cdots, N_j), \ t \equiv (t_1, t_2, \cdots, t_m)^T. \)

3.2 Dual problem

The dual problem of (5) is as follows:

Maximize \( d(w) = \delta_0 \prod_{j=0}^{m} \prod_{k=1}^{N_j} \left( \frac{c_{jk} w_{jk}}{w_{jk}} \right)^{\alpha_{j,k} w_{jk}} \) \hspace{1cm} (3.2)

Subject to \( \sum_{k=1}^{N_j} \delta_{j,k} w_{jk} = \delta_0 \) (normality condition) \( \sum_{j=1}^{m} \sum_{k=1}^{N_j} \delta_{j,k} \alpha_{j,k} w_{jk} = 0, \ (i = 1, 2, \cdots, n) \) (orthogonality condition) where \( \delta_{j} = 1 \ (j = 2, \cdots, m), \ \delta_{j,k} = 1 \ (j = 1, 2, \cdots, m; \ k = 1, 2, \cdots, N_j), \) and \( w_{j0} = 1. \ \delta_{j} = +1, -1 \) and non-negativity conditions, \( w_{j0} = \delta_0 \sum_{k=1}^{N_j} \delta_{j,k} w_{jk} \geq 0, \ \delta_{j,k} \geq 0, \ (j = 1, 2, \cdots, m; k = 1, 2, \cdots, N_j) \) and \( w_{j0} = 1. \)

4. Fuzzy geometric programming \( (FGP) \)

\[ \overline{\text{Minimize}} g_0(t) \] \hspace{1cm} (4.1)

Subject to \( g_j(t) \leq b_j, \ (j = 1, 2, \cdots, m), \ t \geq 0 \)

Here, the symbol “\( \overline{\text{Minimize}} \)” denotes a relaxed or fuzzy version of “Minimize”. Similarly, the symbol “\( \leq \)” denotes a fuzzy version of “\( \leq \)”. There fuzzy requirements
may be quantified by eliciting membership functions \( \mu_j(g_j(t)) \) \((j = 0, 1, 2, \ldots, m)\) from the decision maker for all functions \( g_j(t) \) \((j = 0, 1, 2, \ldots, m)\). By taking account of the rate of increased membership satisfaction, the decision maker must determine the subjective membership function \( \mu_j(g_j(t)) \). It is, in general, a strictly monotone decreasing linear or nonlinear function \( u_j(g_j(t)) \) with respect to \( g_j(t) \) \((j = 0, 1, 2, \ldots, m)\). Here for simplicity, linear membership functions are considered. The linear membership functions can be represented as follows:

\[
\mu_j(g_j(t)) = \begin{cases} 
1, & \text{if } g_j(t) \leq g_j^0 \\
\left(g_j^0 - g_j(t)\right)/\left(g_j^0 - g_j^i\right), & \text{if } g_j^0 \leq g_j(t) \leq g_j^i \\
0, & \text{if } g_j(t) \geq g_j^i 
\end{cases}
\]

for \( j = 0, 1, 2, \ldots, m \).

As shown in Figure 1.

\( g_j^0 \) is the value of \( g_j(t) \) such that the grade of membership function \( \mu_j(g_j(t)) \) is 1.
\( g_j^i \) is the value of \( g_j(t) \) such that the grade of membership function \( \mu_j(g_j(t)) \) is 0.
\( \bar{g}_j \) is the intermediate value of \( g_j(t) \) between \( g_j^0 \) and \( g_j^i \) (i.e. \( \bar{g}_j \in (g_j^0, g_j^i) \)) such that the grade of membership function \( \alpha \in (0, 1) \).

The Problem (4.1) reduces to FGEP when \( g_0(t) \) and \( g_1(t) \) are signomial and osynomial functions.

Based on fuzzy decision making of Bellman and Zadeh [14], we may write

(i)  
\[
\mu_0(t^*) = \max \left( \min \mu_j(g_j(t)) \right) \text{(max - min operator)} 
\]
subject to \( \mu_j(g_j(t)) = \left(g_j^0 - g_j(t)\right)/\left(g_j^0 - g_j^i\right) \) \((j = 0, 1, 2, \ldots, m)\), \( t > 0 \)

(ii)  
\[
\mu_0(t^*) = \max \left( \sum_{j=0}^{m} \lambda_j \mu_j(g_j(t)) \right) \text{(max - additive operator)} 
\]
subject to \( \mu_j(g_j(t)) = \left(g_j^0 - g_j(t)\right)/\left(g_j^0 - g_j^i\right) \) \((j = 0, 1, 2, \ldots, m)\), \( t > 0 \)

(iii)
\[ \mu_D(t^*) = \max \left( \prod_{j=0}^{\infty} \left( \frac{\mu_j(g_j(t))}{(g_j'(t) - g_j(t))} \right) \right) \] (max-product operator) \hspace{1cm} (4.4) \\

subject to \( \mu_j(g_j(t)) = \frac{(g_j'(t) - g_j(t))}{(g_j' - g_j)} \) (\( j = 0, 1, 2, \cdots, m \)), \( t > 0 \)

Here, for \( \lambda_j \) (\( j = 0, 1, 2, \cdots, m \)) are numerical weights considered by decision makers.

For normalized weights \( \sum_{j=0}^{m} \lambda_j = 1 \) and \( \lambda_j \in [0,1] \).

For equal importance of objective and constraint goals, \( \lambda_j = 1 \).

Unlike GP, FGP in general has not been widely circulated in the literature. In 1990, Vermal [16] studied a new concept to use the GP technique for multi-objective fuzzy decision-making problems. He projected the very importance on the product operator, which reduces the DD with a considerable amount. Biswal [13] applied the fuzzy programming technique to solve a multi-objective GP problem as a vector minimization problem. A vector maximization problem can be transformed into a vector minimization problem. Cao [3-4] discussed the properties of a kind of posynomial GP with an LR fuzzy coefficient in objective and constraints. In the sequel, Cao [2] published an important book on FGP, which was the most recent book until now.

If \( g_j(t) \) (\( j = 0, 12, \cdots, m \)) be posynomial function as \( g_j(t) = \sum_{k=1}^{N_j} c_{jk} \sum_{l=1}^{n_j} t_{kl}^{a_{jl}} \) (\( c_{jk} \geq 0 \)) and \( \alpha_{jl} \) (\( i = 1, 2, \cdots, n \); \( k = 1, 2, \cdots, N_j \); \( j = 0, 1, 2, \cdots, m \)) then

(i) max-min operator (4.2) reduces to

\[ \text{Maximize} \left( \frac{\tilde{\lambda}_j(g_j'(t) - g_j(t))}{(g_j' - g_j)} \right) \] \hspace{1cm} (4.5)

Subject to, \( \tilde{\lambda}_j(g_j'(t) - g_j(t)) \geq \tilde{\lambda}_j(g_j'(t) - g_j(t)) \) (\( r = 0, 1, 2, \cdots, m \) and \( r \neq j \)), \( t > 0 \).

So \( V^+_D(t^*) = \tilde{\lambda}_j \left( \frac{g_j'(t) - g_j(t)}{(g_j' - g_j)} \right) V^+ (t^*) \) Where \( t^* \) is obtained by solving the following signomial GP:

\[ \text{Minimize} \ V(t) = \sum_{k=1}^{N_j} c_{jk} \sum_{l=1}^{n_j} t_{kl}^{a_{jl}} \] \hspace{1cm} (4.6)

Subject to, \( \frac{\tilde{\lambda}_j(g_j'(t) - g_j(t))}{(g_j' - g_j)} \sum_{k=1}^{N_j} c_{jk} \sum_{l=1}^{n_j} t_{kl}^{a_{jl}} - \frac{\tilde{\lambda}_j(g_j'(t) - g_j(t))}{(g_j' - g_j)} \sum_{k=1}^{N_j} c_{jk} \sum_{l=1}^{n_j} t_{kl}^{a_{jl}}, \) (\( r = 0, 1, 2, \cdots, m \) and \( r \neq j \)), \( t > 0 \)

(ii) max-additive operator (4.3) reduces to

\[ \text{Maximize} \ V_A(t) = \sum_{j=0}^{\infty} \left( \frac{\tilde{\lambda}_j(g_j'(t) - g_j(t))}{(g_j' - g_j)} \right) \] \hspace{1cm} (4.7)

Subject to \( t > 0 \)
Fuzzy Reliability Optimization Based on Fuzzy Geometric Programming
Method Using Different Operators

So the optimal decision variable \( t^* \) with the optimal objective value \( V^*(t^*) \) can be obtained by \( V^*(t^*) = \sum_{j=0}^{m} \hat{\lambda}_j f_j' - g_j' - U^*(t) \) where \( t^* \) is the optimal decision variable of the unconstrained geometric programming problem

\[
\text{Minimize } U(t) = \sum_{j=0}^{m} \left( \hat{\lambda}_j \left( f_j' - g_j' \right) \right) \sum_{k=1}^{N} C_{jk} \prod_{i=1}^{n} \xi_i^{r_{i,j}} \quad (4.8)
\]

Subject to \( t > 0 \).

(iii) Similarly, Eq. (4.4) can be solved by \( GP \) based on a suitable transformation.

4.1. Numerical example

The series system of reliability has three subsystems with reliability \( R_i \) \((i=1,2,3)\) for the \( i^{th} \) sub system. For the series system the system reliability \( R = R_1 R_2 R_3 \).

Now we have to find the maximization of \( R_1 R_2 R_3 \) subject to the limited available cost. Therefore the problem in fuzzy environment becomes

\[
\text{Max} R_1 R_2 R_3 = R_1 R_2 R_3 \geq 0.5504467 \quad (\text{with tolerance } 0.3102683) \quad (4.9)
\]

s.t. \( 130R_1^{0.94} + 140R_2^{0.91} + 150R_3^{0.89} + \leq 401 \quad (\text{with tolerance } 51) \)

Here, the membership functions for the fuzzy objective and constraint goals are \( \mu_R = (R_1 R_2 R_3 - (0.8607150 - 0.3102683))/0.3102683 = (R_1 R_2 R_3 - 0.5504467)/0.3102683 \)

\( \mu_C = (401 - (130R_1^{0.94}, 140R_2^{0.91}, 150R_3^{0.89}))/51 \)

(i) Based on max-min operator (4.2), \( FGP \) (4.9) reduces to Max \( \mu_R \)

Max \( \text{Max} \left( \left( R_1 R_2 R_3 - 0.7758517 \right)/0.0705803 \right) \)

s.t. \( 385 - (125R_1^{0.96}, 135R_2^{0.895}, 145R_3^{0.87}) \)/10 \( \geq (R_1 R_2 R_3 - 0.7758517)/0.0705803 \) \( \leq R_1, R_2, R_3 \leq 1 \)

i.e. \( \text{Max} 14.16826 R_1 R_2 R_3 - 10.99247 \)

\( s.t. 3.85 - (12.5R_1^{0.96}, 13.5R_2^{0.895}, 14.5R_3^{0.87}) \geq 14.16826 R_1 R_2 R_3 - 10.99247 \) i.e. 49.49247

\( \geq 14.16826 R_1 R_2 R_3 + (12.5R_1^{0.96}, 13.5R_2^{0.895}, 14.5R_3^{0.87}) \) i.e. \( 14.16826 R_1 R_2 R_3 \left( 12.5R_1^{0.96}, 13.5R_2^{0.895}, 14.5R_3^{0.87} \right) \leq 49.49247 \) i.e. \( (14.16826/49.49247) R_1 R_2 R_3, (12.5/49.49247) R_1^{0.96}, (13.5/49.49247) R_2^{0.895}, (14.5/49.49247) R_3^{0.87} \leq 1 \)

To solve equation (4.11), we are to solve the following crisp \( GP \) : Min\( l/14.16826 \)

\( l^{-1} R_1^{-1} R_2^{-1} R_3^{-1} \) s.t. \( 0.2525637 R_1^{0.96}, 0.2727688 R_2^{0.895}, 0.2929739 R_3^{0.87} + 0.286271 R_1 R_2 R_3 \leq 1 \).

\( 0 < R_1, R_2, R_3 \leq 1 \). For this problem \( DD = 5 - (3+1) = 1 \). The dual problem (\( DD \)) of this \( GP \) is Max \( d(W) = (1/14.16826W_1)^{0.96} \cdot (0.2525637/W_1)^{0.895} \cdot (0.2727688/W_2)^{0.87} \).


Based on the max-additive operator (4.3), have
\[ R = W_1 + W_2 + W_3 + W_4 \text{ s.t. } W_{01} = 1, -W_{01} + 0.96W_1 + W_2 = 0, -W_{01} + 0.895W_2 + W_3 = 0, -W_{01} + 0.87W_3 + W_4 = 0. \]
So, \[ W_1 = 1 - W_4 / 0.96, \quad W_2 = 1 - W_4 / 0.895, \quad W_3 = 1 - W_4 / 0.87. \]

Max \( d(W) = 0.0705803 \left( \frac{0.2525637}{(1 - W_4)/0.96} \right) \left( \frac{0.2727688}{(1 - W_4)/0.895} \right) \left[ \left\{ 1/0.96, 1/0.96, 1/0.895 - (1/0.96, 1/0.96, 1/0.895, 1/0.87 - 1) \right\} W_4 \right] \]
For optimality of \( d(W) \), we have \( \frac{\partial \log d(W)}{\partial W_4} = 0 \). That is \( 2.30841 + \log \left( \frac{0.2525637}{W_4} \right) = (1/0.96) \log \left( \frac{0.2727688}{(1 - W_4)} \right) \), \( (1/0.895) \log \left( \frac{0.2441281}{(1 - W_4)} \right) \), \( (1/0.87) \log \left( \frac{0.2548873}{1 - W_4} \right) \).

**Table – 1. Optimal solution of (4.9) by max-min operator (4.2)**

<table>
<thead>
<tr>
<th>( d^* )</th>
<th>0.02990114</th>
<th>( R_i^* )</th>
<th>0.9245300</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_{11}^* )</td>
<td>0.1768750</td>
<td>( R_i^* )</td>
<td>0.9743700</td>
</tr>
<tr>
<td>( W_{12}^* )</td>
<td>0.1897207</td>
<td>( R_i^* )</td>
<td>0.9465400</td>
</tr>
<tr>
<td>( W_{13}^* )</td>
<td>0.1951724</td>
<td>( R_i^* )</td>
<td>0.8526757</td>
</tr>
<tr>
<td>( W_{14}^* )</td>
<td>0.8302027</td>
<td>--</td>
<td>--</td>
</tr>
</tbody>
</table>

It is clear from the above table that the system reliability of series system with objective and cost constraints in fuzzy environment by max-min operator is 0.8526757 which shows that the FGP technique is effective in achieving a superior compromise.

(ii) Based on the max-additive operator (4.3), FGP (4.9) reduces to Max \( \mu_R + \mu_C \), s.t. \( 0 \leq R_i \leq 1 \), Max \( A(R_1, R_2, R_3) = (R_1 R_2 R_3 - 0.5332254) / 0.3274896 \), \( (401 - \{130 R_{10.94}^{0.94}, 140 R_{20.94}^{0.94}, 150 R_{30.94}^{0.89}\}) / 41 \) s.t. \( 0 \leq R_i \leq 1 \). Here, \( A(R_1, R_2, R_3) = (1/0.3274896) R_1 R_2 R_3 - (1/41) \{130 R_{10.94}^{0.94}, 140 R_{20.94}^{0.94}, 150 R_{30.94}^{0.89}\} + (\{401/41\} - (0.5332254/0.3274896)) - A(R_1, R_2, R_3) = (130/41) R_{10.94}, (140/41) R_{20.94}, (150/41) R_{30.89} - 3.053532 R_1 R_2 R_3 = 8.152267 A(R_1, R_2, R_3) - 8.152267 At first we take the problem Min \( (130/41) R_{10.94}, (140/41) R_{20.94}, (150/41) R_{30.89} - 3.053532 R_1 R_2 R_3 \) s.t. \( 0 \leq R_1, R_2, R_3 \leq 1 \). For this problem \( DD = 4 - (3+1) = 0 \). The dual problem \( DP \) of this G.P is Max \( d(W) = \{130/41 W_1, (140/41 W_2) \}^{W_1}, (150/41 W_3) \}^{W_2}, (3.053532 W_4) \}^{W_3} \) s.t. \( W_1 + W_2 + W_3 - W_4 = 1 \), \( 0.94W_1 - W_4 = 0 \), \( 0.91W_2 - W_4 = 0 \), \( 0.89W_3 - W_4 = 0 \)

**Table – 2. Optimal solution of (4.9) by max-additive operator (4.3)**

| \( d(W^*) \) | 7.184581 | \( R_i^* \) | 0.95321000 |
It is clear from the above table that the system reliability of series system with objective and cost constraints in fuzzy environment by max-additive operator is 0.8569501 which shows that the FGP technique is effective in achieving a superior compromise.

(iii) Based on the max-product operator (4.4), FGP reduces to Max $\mu_R \cdot \mu_c$.

Max $\left( \left( R_1 R_2 R_3 - 0.7758517 \right) / 0.0705803 \right) \cdot \left( 385 - \left( 125R_1^{0.06}, 135R_2^{0.895}, 145R_3^{0.87} \right) \right)$

$0 \leq R_1, R_2, R_3 \leq 1$,

Max $f(R_1, R_2, R_3) = 545.4780R_1R_2R_3 - 177.1032R_1^{0.96}R_2R_3 - 191.2715R_2^{0.895}R_3 - 205.4398R_1R_2^{0.87} + 137.4059R_1^{0.06} + 148.3983R_2^{0.895} + 159.3908R_3^{0.87}$

Let $f(R_1, R_2, R_3) = -f(R_1, R_2, R_3) - 423.21$ Min $f(R_1, R_2, R_3) = 545.4780R_1R_2R_3 + 177.1032R_1^{0.96}R_2R_3 + 191.2715R_1R_2^{0.87} + 205.4398R_1R_2R_3^{0.87} - 137.4059R_1^{0.06} - 148.3983R_2^{0.895} - 159.3908R_3^{0.87}$

The dual problem (DP) of this GP is Max $d$(W)

$\xi \left( \frac{545.4780}{W_1} \right)^W_1 \cdot \left( \frac{177.1032}{W_2} \right)^W_2 \cdot \left( \frac{191.2715}{W_3} \right)^W_3 \cdot \left( \frac{205.4398}{W_4} \right)^W_4$, $\sum^W_1 \xi = 1$ or $-1$, Let $\xi = -1$ s.t.

$-W_1 + W_2 + W_3 - W_4 - W_5 - W_6 = \xi - W_1 + 1.96W_2 + W_3 + W_4 - 0.96W_5 = 0 -W_1 + W_2 + 1.895W_1 + W_4 - 0.895W_6 = 0 -W_1 + W_2 + W_3 + W_4 + 1.87W_1 - 0.87W_5 = 0$

Again, $545.4780R_1R_2R_3 = W_1 \cdot d$, $177.1032R_1^{0.96}R_2R_3 = W_2 \cdot d$, $191.2715R_1R_2^{0.87} = W_3 \cdot d$, $205.4398R_1R_2R_3^{0.87} = W_4 \cdot d$,

$137.4059R_1^{0.06} = W_5 \cdot d$, $148.3983R_2^{0.895} = W_6 \cdot d$, $159.3908R_3^{0.87} = W_7 d$

Table –3. Optimal solution of (4.9) by max-product operator (4.4)

<table>
<thead>
<tr>
<th>$d^*$</th>
<th>$W_{1}^*$</th>
<th>$W_{2}^*$</th>
<th>$W_{3}^*$</th>
<th>$W_{4}^*$</th>
<th>$W_{5}^*$</th>
<th>$W_{6}^*$</th>
<th>$W_{7}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.280157</td>
<td>0.9999992</td>
<td>0.2875887</td>
<td>0.4295827</td>
<td>0.2828280</td>
<td>0.2875900</td>
<td>0.4295830</td>
<td>0.2828283</td>
</tr>
</tbody>
</table>

It is clear from the above table that the system reliability of series system with objective and cost constraints in fuzzy environment by max-product operator is 0.8486808 which shows that the FGP technique is effective in achieving a superior compromise.
5. Conclusion

Here we have dealt with series system reliability optimization problem with objective and cost constraints in fuzzy environment. Numerical examples are solved by different operators like max-min operator, max-additive operator and max-product operators to show that the FGP technique is effective in achieving a superior compromise between mutually conflicting objective and constraints when the problem parameters, prescribed goal and design constraint are not known precisely. It is hoped that this technique can be used in many practical engineering model on reliability optimization.

References

Concept Analysis with Interior and Closure Operators

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Abstract: The study of concept analysis is done within the lattice structure. In this paper we are interested to study concept analysis in some different way. Here we will introduce some algebraic operators and study some algebra. Also we will study different types of Information-System and the rules of decision making using this new type of concept analysis.

Key words: Formal Concept Analysis, Semi-Lattices, Fuzzy Set, Information System.

1. Introduction

Formal concept analysis or context analysis and study of information system is almost similar thing. In this paper we are interested to study concept analysis or context analysis as a different study from the study of the information system.

In your real life if we take any concept (say about certain disease), then there are some attribute associated with this concept. An association of the concepts with the attributes we say a context.

In this paper we will introduce two operators-interior and closure operators and will study some algebra.

Lastly we will study different types of Information Systems (IS).

2. Context

In our study all the sets are assumed to be finite.

Definition 2.1. A context $C$ is a triplet $(L,V,v)$, where $L$ is a Lattice-called concept lattice, $V$ is the set of attributes and $v:L \rightarrow 2^{V}$ called valuation mapping satisfying the following conditions: for any $\alpha, \beta \in L$
If $L$ is a join (or meet) semi-lattice and with conditions (i), (ii) and (iii) (or (iv)) $(L, V, v)$ is called join (resp. meet) semi-context.

**Theorem 2.2.** Let be $(L, V, v)$ a context, for any $X \subseteq V$ we define $F_X : L \rightarrow [0, 1]$ as

$$F_X (\alpha) = \frac{\|v(\alpha) \cap X\|}{\|v(\alpha)\|}.$$  

$F_X$ is a fuzzy subset of $L$.

**Definition 2.3.** Let $(L, V, v)$ be a context, for any $\emptyset \neq A \subseteq L$ we define fuzzy relation $\Delta_A : \wp(V) \times \wp(V) \rightarrow [0, 1]$ on $\wp(V)$ defined as $\Delta_A (X, Y) = \sum_{\alpha \in A} F_X (\alpha) - F_Y (\alpha) \|A\|$ called the distance measure between $X$ and $Y$ in $\wp(V)$ with respect to $A$.

The function $S_A : \wp(V) \times \wp(V) \rightarrow [0, 1]$ defined as $S_A (X, Y) = 1 - \Delta_A (X, Y)$ called the similarity measure between $X$ and $Y$ in $\wp(V)$ with respect to $A$.

**Theorem 2.4.** $\Delta_A$ is a pseudo-metric on $\wp(V)$. Also for any $X, Y \subseteq V$, $S_A (X, Y) = S_A (Y, X)$.

**Proof.** It is not very difficult to see that for any $X, Y, Z \subseteq V$,

(i) $\Delta_A (X, X) = 0$

(ii) $\Delta_A (X, Y) = \Delta_A (Y, X)$

(iii) $\Delta_A (X, Z) \leq \Delta_A (X, Y) + \Delta_A (Y, Z)$.

Thus $\Delta_A$ is a pseudo-metric on $\wp(V)$.

Other part is straight forward.

**Definition 2.5.** For any $A \subseteq L$ we define a relation $=_{\Delta_A}$ on $\wp(V)$ defined as: $X =_{\Delta_A} Y$ iff $\Delta_A (X, Y) = 0$ or equivalently $S_A (X, Y) = 1$.

Now we have the following easy consequence:

**Theorem 2.6.** $=_{\Delta_A}$ is an equivalence relation on $\wp(V)$.

**Definition 2.7.** Let $(L, V, v)$ be a context, we define $I : \wp(V) \rightarrow \wp(L)$ and $C : \wp(V) \rightarrow \wp(L)$ called respectively interior operator and closure operator defined as:
\[ I(X) = \{ \alpha \in L : v(\alpha) \subseteq X \}, \quad C(X) = \{ \alpha \in L : v(\alpha) \cap X \neq \emptyset \}. \]

\[ \partial X = C(X) - I(X) \] is called the boundary region of \( X \) and \( E(X) = L - C(X) \) is called the exterior region of \( X \).

**Theorem 2.8.** Let \((L, V, v)\) be a context, for any \( X \subseteq V \)

(i) \( I(X) = \{ \alpha \in L : F_X(\alpha) = 1 \} \)

(ii) \( C(X) = \{ \alpha \in L : F_X(\alpha) > 0 \} \)

(iii) \( \partial(X) = \{ \alpha \in L : 0 < F_X(\alpha) < 1 \} \)

(iv) \( E(X) = \{ \alpha \in L : F_X(\alpha) = 0 \} \).

**Definition 2.9.** Let \((L, V, v)\) be a context. We define three relations \( =_I, =_C \) and \( = \) on \( \wp(V) \) as: \( X =_I Y \) iff \( I(X) = I(Y) \), \( X =_C Y \) iff \( C(X) = C(Y) \) and \( X = Y \) iff \( I(X) = I(Y) \) and \( C(X) = C(Y) \).

Now we have the following easy consequences:

**Theorem 2.10.** \( =_I, =_C \) and \( = \) are all equivalence relation on \( \wp(V) \). Let for any \( X \subseteq V \), \([X]_I\), \([X]_C\) and \([X]_\circ\) be the equivalence classes of \( X \) for the equivalence relations \( =_I, =_C \) and \( = \) respectively. Let \( \wp(V)/=_I, \wp(V)/=_C \) and \( \wp(V)/= \) be the collection of equivalence classes for the equivalence relations \( =_I, =_C \) and \( = \) respectively.

**Theorem 2.11.** For a context \((L, V, v)\) for any \( X, Y \subseteq V \)

(i) \( X \subseteq Y \Rightarrow I(X) \subseteq I(Y) \) and \( C(X) \subseteq C(Y) \).

(ii) \( I(X \cap Y) = I(X) \cap I(Y) \).

(iii) \( C(X \cup Y) = C(X) \cup C(Y) \).

(iv) \( I(V - X) = L - C(X) \).

(v) \( C(V - X) = L - I(X) \).

(vi) \( I(\emptyset) = C(\emptyset) = \emptyset, \; I(V) = C(V) = L \).

**Proof.** (i) and (vi) directly follows from the definitions.

(ii) \( X \cap Y \subseteq X \) and \( X \cap Y \subseteq Y \), so \( I(X \cap Y) \subseteq I(X) \) and \( I(X \cap Y) \subseteq I(Y) \) (by (i), thus \( I(X \cap Y) \subseteq I(X) \cap I(Y) \). To prove the reverse inequality, let \( \alpha \in I(X) \cap I(Y) \), i.e. \( \alpha \in I(X) \) and \( \alpha \in I(Y) \), so \( v(\alpha) \subseteq X \) and \( v(\alpha) \subseteq Y \), thus \( v(\alpha) \subseteq X \cap Y \) and hence \( \alpha \in I(X \cap Y) \), so \( I(X) \cap I(Y) \subseteq I(X \cap Y) \). Hence \( I(X \cap Y) = I(X) \cap I(Y) \).
(iii) Again using (i) we have \( C(X) \cup C(Y) \subseteq C(X \cup Y) \). To prove the reverse inequality let \( \alpha \in C(X \cup Y) \), so \( v(\alpha) \cap (X \cup Y) \neq \emptyset \), so \( v(\alpha) \cap X \neq \emptyset \) or \( v(\alpha) \cap Y \neq \emptyset \), i.e. \( \alpha \in C(X) \) or \( \alpha \in C(Y) \), so \( \alpha \in C(X) \cup C(Y) \). Hence \( C(X \cup Y) \subseteq C(Y) \cup C(Y) \). Thus we have \( C(X \cup Y) = C(X) \cup C(Y) \).

(iv) Let \( \alpha \in I(V - X) \iff v(\alpha) \subseteq (V - X) \), \( \iff v(\alpha) \cap X = \emptyset \iff \alpha \notin C(X) \iff \alpha \in (L - C(X)) \). Thus \( I(V - X) = L - C(X) \).

(v) Proof is similar to proof of (iv).

Using the above Theorem 2.11 we can define the following well defined operators:

**Definition 2.12.** We define the following operators:

(i) \( \cap \) on \( \phi(V)/=_{j} \) as \( [X]_{j} \cap [Y]_{j} = [X \cap Y]_{j} \),

(ii) \( \cup \) on \( \phi(V)/=_{c} \) as \( [X]_{c} \cup [Y]_{c} = [X \cup Y]_{c} \),

(iii) \( \rightarrow \) on \( \phi(V)/= \) as \( \neg [X]_{e} = [V - X]_{e} \).

Again using the above Theorem 2.11 we can define the following well defined inequalities:

**Definition 2.13.** We define the following operators:

(i) \( \leq_{j} \) on \( \phi(V)/=_{j} \) as \( [X]_{j} \leq [Y]_{j} \) iff \( I(X) \subseteq I(Y) \).

(ii) \( \leq_{c} \) on \( \phi(V)/=_{c} \) as \( [X]_{c} \leq [Y]_{c} \) iff \( C(X) \subseteq C(Y) \).

(iii) \( \equiv \) on \( \phi(V)/= \) as \( [X]_{e} \equiv [Y]_{e} \) iff \( I(X) \subseteq I(Y) \) and \( C(X) \subseteq C(Y) \).

**Theorem 2.14.** (i) \( (\phi(V)/=_{j}, \leq_{j}) \) is a partial order set with least element \( [\emptyset]_{j} \), and greatest element \( [V]_{j} \).

(ii) \( (\phi(V)/=_{c}, \leq_{c}) \) is a partial order set with least element \( [\emptyset]_{c} \) and greatest element \( [V]_{c} \).

(iii) \( (\phi(V)/=, \equiv) \) is a partial order set with least element \( [\emptyset]_{e} \) and greatest element \( [V]_{e} \) is a complemented partial order set.

**Proof.** We are going to prove this theorem for (i) only, proof of others are similar.

(i) A relation is a partial order relation if it is reflexive, transitive and antisymmetric. It is not much difficult to see that the relation \( \leq_{j} \) is a partial order relation on \( \phi(V)/=_{j} \). Also it is easy to see that the partial order set \( (\phi(V)/=_{j}, \leq_{j}) \) possess the least and greatest elements which are none other than \( [\emptyset]_{j} \) and \( [V]_{j} \) respectively. A partial order set is meet-semilatice if it is closed under finite meet, i.e. any two element have their infimum with respect to the partial order relation.
A lattice is simultaneously a meet-semilattice and join-semilattice both. Now we have the following consequences:

**Theorem 2.15.** The mappings:

(i) \( f_i : (L, \leq) \rightarrow \left( \mathcal{P}(V)/\mathcal{I}, \subseteq \right) \) defined as \( f_i(\alpha) = \left[ v(\alpha) \right]_{\mathcal{I}} \) is a meet-semilattice homomorphism.

(ii) \( f_c : (L, \leq) \rightarrow \left( \mathcal{P}(V)/\mathcal{C}, \subseteq \right) \) defined as \( f_c(\alpha) = \left[ v(\alpha) \right]_{\mathcal{C}} \) is a join-semilattice homomorphism.

(iii) \( f : (L, \leq) \rightarrow \left( \mathcal{P}(V)/\mathcal{I}, \subseteq \right) \) defined as \( f(\alpha) = \left[ v(\alpha) \right]_{\mathcal{I}} \) is a partial order set homomorphism.

**Proof.** We are going to prove this theorem for (i) only, the proof of others are similar.

(i) For \( \alpha, \beta \in L \) , \( \alpha \leq \beta \Rightarrow v(\alpha) \subseteq v(\beta) \Rightarrow I(v(\alpha)) \subseteq I(v(\beta)) \Rightarrow \left[ v(\alpha) \right]_{\mathcal{I}} \subseteq \left[ v(\beta) \right]_{\mathcal{I}} \), thus \( f_i \) is a partial order homomorphism.

Again \( f_i(\alpha \land \beta) = \left[ v(\alpha \land \beta) \right]_{\mathcal{I}} = \left[ v(\alpha) \cap v(\beta) \right]_{\mathcal{I}} = \left[ v(\alpha) \right]_{\mathcal{I}} \cap \left[ v(\beta) \right]_{\mathcal{I}} \), thus \( f_i \) is a meet-semilattice homomorphism.

3. Information system

In this section we will study different types of Information-Systems and some methods of decision making in such information-system. In this section we consider all the sets unless otherwise stated are assumed to be finite.

**Definition 3.1.** An information system (IS) \( \mathcal{I} \) is a pair \( (C, O) \), where \( C = (L, V, v) \) is a context and \( O \) is a set of objects. For each \( a \in V \) , we associate a mapping \( a : O \rightarrow \{0, 1\} \). If \( a(x) = 1 \) we say the object \( x \) is associated with the attribute \( a \), else we say \( x \) is not associated with the attribute \( a \).

Let \( U \subseteq O \) , we define \( I(U) = \{ a \in V : a(x) = 1, \forall x \in U \} \), \( C(U) = \{ a \in V : a(x) = 1, \exists x \in U \} \) and \( \partial(U) = C(U) - I(U) \). It is easy to see that \( I(U) \subseteq C(U) \).

We say a concept \( \alpha \) is

(i) Strongly associate with \( U \) if \( v(\alpha) \subseteq I(U) \), i.e. \( F_{I(U)}(\alpha) = 1 \). We write this situation as \( U \triangleright^s \alpha \).

(ii) Weakly associate with \( U \) if \( v(\alpha) \subseteq C(U) \), i.e. \( F_{C(U)}(\alpha) = 1 \). We write this situation as \( U \triangleright^w \alpha \).

(iii) Associate with \( U \) at strong \( e \)-level if \( F_{I(U)}(\alpha) \geq e \), where \( 0 < e \leq 1 \). We write this situation as \( U \triangleright^e \alpha \).
(iv) Associate with $U$ at weak $\varepsilon$-level if $F_{c(U)}(a) \geq \varepsilon$, where $0 < \varepsilon \leq 1$. We write this situation as $U \triangleright_{\varepsilon} a$.

Now we have the following easy consequences:

**Theorem 3.2.** Let $\mathcal{I} = (C, O)$, be an information system, let $U, U_1, U_2 \subseteq O$, $\alpha, \beta \in L$:

(a) $U \triangleright^{s} \alpha \Rightarrow U \triangleright^{s} \alpha$, $U \triangleright^{e}_{\varepsilon} \alpha \Rightarrow U \triangleright^{e}_{\varepsilon} \alpha$

(b) For $\triangleright^{s} \Rightarrow \triangleright^{e}$:

(i) $U \triangleright \alpha$, $\beta \leq \alpha \Rightarrow U \triangleright \beta$

(ii) $U_1 \triangleright \alpha$, $U_2 \triangleright \beta \Rightarrow U_1 \cup U_2 \triangleright \alpha \lor \beta$

(c) For $\triangleright^{s} \Rightarrow \triangleright^{e}$, $0 < \varepsilon \leq 1$:

(i) $U_1 \triangleright \alpha$, $U_1 \subseteq U_2 \Rightarrow U_2 \triangleright \alpha$.

**Example 3.3.** We consider the IS $\mathcal{I} = (C, O)$ with $C = (L, V, v)$ as: $L = \{a_1, a_2, a_3\}$, $V = \{a_1, a_2, a_3, a_4\}$, $v(a_1) = \{a_1, a_2\}$, $v(a_2) = \{a_2, a_3\}$, $v(a_3) = \{a_3, a_4\}$ and $O = \{x_1, x_2, x_3, x_4, x_5\}$ with:

$$
\begin{array}{cccc}
a_1 & a_2 & a_3 & a_4 \\
x_1 & 0 & 1 & 1 & 1 \\
x_2 & 0 & 0 & 1 & 1 \\
x_3 & 1 & 0 & 1 & 1 \\
x_4 & 1 & 1 & 0 & 0 \\
x_5 & 1 & 1 & 0 & 0 \\
\end{array}
$$

$U = \{x_2, x_3\}$, $I(U) = \{a_3, a_4\}$, $C(U) = \{a_1, a_3, a_4\}$ and $\partial(U) = \{a_1\}$. So, $\alpha_3 \triangleright^{s}_{0.4} U$, $\alpha_5 \triangleright^{e} U$.

Let us now generalize IS in fuzzy setting as:

**Definition 3.4.** An information system with fuzzy values (ISFV) $\mathcal{I}$ is a pair $(C, O)$, where $C = (L, V, v)$ is a context and $O$ is a set of objects, for each $a \in V$, we associate a mapping $a : O \rightarrow [0, 1]$ called object gradation associated with $a$, which is a fuzzy subset of $O$, denoted by the same symbol $a$.

For $U \subseteq O$ and $0 < \varepsilon \leq 1$ we define $I_{\varepsilon}(U) = \{a \in V : \forall x \in U, a(x) \geq \varepsilon\}$, $C_{\varepsilon}(U) = \{a \in V : \exists x \in U, a(x) \geq \varepsilon\}$ and $\partial_{\varepsilon}(U) = C_{\varepsilon}(U) - I_{\varepsilon}(U)$. Obviously $I_{\varepsilon}(U) \subseteq C_{\varepsilon}(U)$. If $U$ is a singleton set $\{x\}$, then $I_{\varepsilon}(U) = C_{\varepsilon}(U) = IC_{\varepsilon}(x)$ (say). For $0 < \varepsilon \leq 1$, $U \subseteq O$, we say $\alpha \in L$ is associated with $U$ at

(i) Strong $\varepsilon$-level if $v(a) \subseteq I_{\varepsilon}(U)$, we write $\alpha \triangleright_{\varepsilon} U$.
(ii) Weak $\varepsilon$-level if $v(\alpha) \subseteq C_\varepsilon(U)$, we write $\alpha \triangleright^w \varepsilon U$.

**Theorem 3.5.** For a ISFV $\mathcal{I} = (\mathcal{C}, O)$ with $\mathcal{C} = (L, V, v)$, for $x \in O$ and $\alpha \in L$, if we define level of $\alpha$ for $x$ as $l_\alpha(x) = \max \{\varepsilon > 0 : v(\alpha) \subseteq IC_\varepsilon(x)\}$ or $l_\alpha(x) = 0$ otherwise; then $l_\alpha$ is a fuzzy subset of $O$ and $l_\alpha(x) = \min \{a(x) : a \in v(\alpha)\}$.

**Theorem 3.6.** For an ISFV $\mathcal{I} = (\mathcal{C}, O)$ with $\mathcal{C} = (L, V, v)$, for $U \subseteq O$,

(i) $\alpha \triangleright^s \varepsilon U \Rightarrow \alpha \triangleright^w \varepsilon U$

(ii) $\alpha \leq \beta \Rightarrow \beta \triangleright U \Rightarrow \alpha \triangleright U$, for $\triangleright = \triangleright^s$, $\triangleright^w$

(iii) $U_1 \subseteq U_2 \subseteq O$, $\alpha \triangleright^s \varepsilon U_1 \Rightarrow \alpha \triangleright^s \varepsilon U_2$ and $\alpha \triangleright^w \varepsilon U_1 \Rightarrow \alpha \triangleright^w \varepsilon U_2$.

**Example 3.7.** We consider the ISFV $\mathcal{I} = (\mathcal{C}, O)$ with $\mathcal{C} = (L, V, v)$ as: $L = \{a_1, a_2, a_3\}$, $V = \{a_1, a_2, a_3, a_4\}$, $v(a_1) = \{a_1, a_2\}$, $v(a_2) = \{a_3\}$, $v(a_3) = \{a_1, a_4\}$ and $O = \{x_1, x_2, x_3, x_4, x_5\}$ with: Object gradation:

\[
\begin{array}{cccc}
  a_1 & a_2 & a_3 & a_4 \\
  x_1 & 0.1 & 0.3 & 0.5 & 1 \\
  x_2 & 0 & 0.2 & 0.5 & 0.6 \\
  x_3 & 0.9 & 0.3 & 0.4 & 0.7 \\
  x_4 & 1 & 0.2 & 0.5 & 0.2 \\
  x_5 & 0.1 & 0.3 & 0.9 & 0.9 \\
\end{array}
\]

$U = \{x_1, x_3\}$, $I_{0.5}(U) = \{a_4\}$, $C_{0.5}(U) = \{a_1, a_2, a_4\}$ and $\partial_{0.5}(U) = \{a_1, a_3\}$. So, $a_3 \triangleright^w_{0.5} U$.

**Definition 3.8.** An information system with fuzzy attributes (ISFA) $\mathcal{I}$ is a pair $(\mathcal{C}, O)$, where $\mathcal{C} = (L, V, v)$ is a context and $O$ is a set of objects, for each $\alpha \in L$, we associate a mapping $\alpha : V \rightarrow [0, 1]$ called value gradation associated with $\alpha$, which is a fuzzy subset of $V$, denoted by the same symbol $\alpha$ and for $\alpha \leq \beta$, $\alpha$ is a fuzzy subset of $\beta$; for each $a \in V$, we associate a mapping $a : O \rightarrow \{0, 1\}$.

We denote the $\varepsilon$-cut, where $0 < \varepsilon \leq 1$ of $\alpha$ as $\sigma_\varepsilon(\alpha) = \{a \in V : \alpha(a) \geq \varepsilon\}$.

For $U \subseteq O$, we say $\alpha \in L$ associated with $U$ at

(i) Strong $\varepsilon$-level if $\sigma_\varepsilon(\alpha) \subseteq I(U)$, we write $\alpha \triangleright^s \varepsilon U$

(ii) Weak $\varepsilon$-level if $\sigma_\varepsilon(\alpha) \subseteq C(U)$, we write $\alpha \triangleright^w \varepsilon U$.

**Theorem 3.9.** For an ISFA $\mathcal{I} = (\mathcal{C}, O)$, with $\mathcal{C} = (L, V, v)$, for $U \subseteq O$,

(i) $\alpha \triangleright^s \varepsilon U \Rightarrow \alpha \triangleright^w \varepsilon U$
Example 3.10. We consider the $ISFA = (C, O)$ with $C = (L, V, v)$ as: $L = \{\alpha_1 < \alpha_2 < \alpha_3\}$, $V = \{a_1, a_2, a_3, a_4\}$ with: Value gradation:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$x_3$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$x_4$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_5$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$\sigma_{\varepsilon, \delta} (\alpha_i) = \{a_1, a_2\}$, $U = \{x_1, x_2\}$, $I(U) = \{a_1, a_2\}$, $C(U) = \{a_1, a_2, a_3, a_4\}$ and $|\partial(U)| = \{a_4\}$. So, $\alpha \triangleright_{0.5} U$.

Definition 3.11. An information system with fuzzy values and fuzzy attributes ($ISFVFA$) $I$ is a pair $(C, O)$, where $C = (L, V, v)$ is a context and $O$ is a set of objects, for each $\alpha \in L$, we associate a mapping $\alpha : V \rightarrow [0,1]$, which is a fuzzy subset of $V$ and for $\alpha \leq \beta$, $\alpha$ is a fuzzy subset of $\beta$; for each $a \in V$, we associate a mapping $a : O \rightarrow [0,1]$, which is a fuzzy subset of $O$.

Let $0 < \varepsilon, \delta \leq 1$. For $U \subseteq O$, we say $\alpha \in L$ associated with $U$ at

(i) Strong $(\varepsilon, \delta)$-level if $\sigma_{\varepsilon} (\alpha) \subseteq I_{\delta} (U)$, we write $U \triangleright_{\varepsilon, \delta} \alpha$,

(ii) Weak $(\varepsilon, \delta)$-level if $\sigma_{\varepsilon} (\alpha) \subseteq C_{\delta} (U)$, we write $U \triangleright_{\varepsilon, \delta} \alpha$.

Theorem 3.12. For an $ISFVFA$ $I = (C, O)$, with $C = (L, V, v)$, for $U \subseteq O$,

(i) $\alpha \triangleright_{\varepsilon, \delta} U \Rightarrow \alpha \triangleright_{\varepsilon, \delta} U$

(ii) $\alpha \leq \beta$, $\beta \triangleright U \Rightarrow \alpha \triangleright U$, for $\Rightarrow \triangleright_{\varepsilon, \delta}$

(iii) $U_1 \subseteq U_2 \subseteq O$, $\alpha \triangleright_{\varepsilon, \delta} U_2 \Rightarrow \alpha \triangleright_{\varepsilon, \delta} U_1$ and $\alpha \triangleright_{\varepsilon, \delta} U_1 \Rightarrow \alpha \triangleright_{\varepsilon, \delta} U_2$.

Example 3.13. We consider the $ISFVFA$ $I = (C, O)$ with $C = (L, V, v)$ as: $L = \{\alpha_1 < \alpha_2 < \alpha_3\}$, $V = \{a_1, a_2, a_3, a_4\}$ with: Object gradation:
Concept Analysis with Interior and Closure Operators

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\[ \sigma_{0.6}(\alpha_3) = \{a_1, a_2\} \quad U = \{x_2, x_4\} \quad \Omega_{0.5}(U) = \{a_2, a_3\} \quad C_{0.5}(U) = \{a_1, a_2, a_3, a_4\} \quad \text{and} \quad \partial_{0.5}(U) = \{a_1, a_4\} \]

So, \( \alpha_3 \triangleright_{0.6,0.5} U \).

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References


On Somewhat Fuzzy Nearly Continuous Functions

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Abstract:
In this paper the concept of somewhat fuzzy nearly continuous functions, somewhat fuzzy nearly open functions are introduced and studied. Besides giving characterizations of these functions, several interesting properties of these functions are also given. Several examples are given to illustrate the concepts introduced in this paper.

Key Words:
Somewhat fuzzy continuous, somewhat fuzzy nearly continuous, somewhat fuzzy open, somewhat fuzzy nearly open, fuzzy irresolute, fuzzy pre semi-open, fuzzy dense, fuzzy nowhere dense, fuzzy Baire space.

1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced by L. A. Zadeh in his classical paper [12] in the year 1965. Thereafter the paper of C. L. Chang [3] in 1968 paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since then much attention has been paid to generalize the basic concepts of General Topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. The notion of continuity is of fundamental importance essentially in almost all branches of Mathematics. Hence it is of considerable significance from applications view point, to formulate and study new variants of fuzzy continuity.

In classical topology, the class of somewhat continuous functions was introduced and studied by Karil. R. Gentry and Hughes B. Hoyle, III [6]. Later, the concept of “somewhat” in classical topology has been extended to fuzzy topological spaces. Somewhat fuzzy continuous functions, somewhat fuzzy open functions on fuzzy topological spaces were introduced and studied by G. Thangaraj and G. Balasubramanian in [9]. The concept of somewhat nearly continuous functions were introduced and
studied in classical topology by Z. Piotrowski [8]. The concept of nearly open functions were introduced and studied extensively in classical topology by D. S. Jankovic and C. Konstadilaki Savvopoulou [5]. In this paper we introduce the concepts of somewhat fuzzy nearly continuous functions, somewhat fuzzy nearly open functions. Besides giving characterizations of these functions, several interesting properties of these functions are also given. Several examples are given to illustrate the concepts introduced in this paper.

2. Preliminaries

Now we introduce some basic notions and results that are used in the sequel. In this work by a fuzzy topological space we shall mean a non-empty set $X$ together with a fuzzy topology $T$ (in the sense of Chang) and denote it by $(X, T)$. The interior, closure and the complement of a fuzzy set $\lambda$ will be denoted by $\text{int}(\lambda)$, $\text{cl}(\lambda)$ and $1-\lambda$ respectively.

**Definition 2.1.** Let $(X, T)$ be a fuzzy topological space and $\lambda$ be a fuzzy set in $(X, T)$. We define $\text{cl}(\lambda) = \bigwedge \{\mu/\lambda \leq \mu, 1-\mu \in T\}$ and $\text{int}(\lambda) = \bigvee \{\mu/\mu \leq \lambda, \mu \in T\}$.

For any fuzzy set in a fuzzy topological space $(X, T)$, it is easy to see that $1-\text{cl}(\lambda) = \text{int}(1-\lambda)$ and $1-\text{int}(\lambda) = \text{cl}(1-\lambda)$ [1].

**Definition 2.2.** Let $(X, T)$ and $(Y, S)$ be any two fuzzy topological spaces. Let $f$ be a function from the fuzzy topological space $(X, T)$ to the fuzzy topological space $(Y, S)$. Let $\lambda$ be a fuzzy set in $(Y, S)$. The inverse image of $\lambda$ under $f$ written as $f^{-1}(\lambda)$ is the fuzzy set in $(X, T)$ defined by $f^{-1}(\lambda)(x) = \lambda(f(x))$ for all $x \in X$. Also the image of $\lambda$ in $(X, T)$ under $f$ written as $f(\lambda)$ is the fuzzy set in $(Y, S)$ defined by

$$f(\lambda)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \lambda(x) & \text{if } f^{-1}(y) \text{ is non-empty;} \\ 0 & \text{otherwise.} \end{cases}$$

for each $y \in Y$.

**Lemma 2.1** [3]. Let $f : (X, T) \rightarrow (Y, S)$ be a mapping. For fuzzy sets $\lambda$ and $\mu$ of $(X, T)$ and $(Y, S)$ respectively, the following statements hold:

1. $f f^{-1}(\mu) \leq \mu$;
2. $f^{-1}f(\lambda) \geq \lambda$;
3. $f(1-\lambda) \geq 1-f(\lambda)$;
4. $f^{-1}(1-\mu) = 1-f^{-1}(\mu)$;
If \( f \) is one-to-one, then \( f^{-1}(\lambda) = \lambda \).

If \( f \) is onto, then \( f f^{-1}(\mu) = \mu \).

If \( f \) is one-to-one and onto, then \( f(1 - \lambda) = 1 - f(\lambda) \).

**Lemma 2.2** [1]. Let \( f : (X, T) \to (Y, S) \) be a mapping and \( \{\lambda_a\} \) be a family of fuzzy sets of \( Y \). Then

(a) \( f^{-1}\left(\bigcup_a \lambda_a\right) = \bigcup_a f^{-1}(\lambda_a) \).

(b) \( f^{-1}\left(\bigcap_a \lambda_a\right) = \bigcap_a f^{-1}(\lambda_a) \).

**Lemma 2.3** [4]. Let \( f : (X, T) \to (Y, S) \) be a mapping and \( \{A_j\}_{j \in J} \) be a family of fuzzy sets in \( X \). Then

(a) \( f\left(\bigcup_{j \in J} A_j\right) = \bigcup_{j \in J} f(A_j) \).

(b) \( f\left(\bigcap_{j \in J} A_j\right) \leq \bigcap_{j \in J} f(A_j) \).

**Definition 2.3** [9]. A fuzzy set \( \lambda \) in a fuzzy topological space \( (X, T) \) is called fuzzy dense if there exists no fuzzy closed set \( \mu \) in \( (X, T) \) such that \( \lambda < \mu < 1 \).

**Definition 2.4** [9]. A fuzzy set \( \lambda \) in a fuzzy topological space \( (X, T) \) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set \( \mu \) in \( (X, T) \) such that \( \mu < \text{cl}(\lambda) \). That is, \( \text{int} \text{cl}(\lambda) = 0 \).

**Definition 2.5** [9]. A fuzzy set \( \lambda \) in a fuzzy topological space \( (X, T) \) is called fuzzy first category if \( \lambda = \bigvee_{i=1}^{\infty} \lambda_i \), where \( \lambda_i \)'s are fuzzy nowhere dense sets in \( (X, T) \). Any other fuzzy set in \( (X, T) \) is said to be of second category.

**Definition 2.6.** A fuzzy set \( \lambda \) in a fuzzy topological space \( (X, T) \) is called:

(a) Fuzzy semi-open [1] if \( \lambda \leq \text{cl int}(\lambda) \).

(b) Fuzzy pre-open [2] if \( \lambda \leq \text{int} \text{cl}(\lambda) \).

**Definition 2.7.** A fuzzy set \( \lambda \) in a fuzzy topological space \( (X, T) \) is called:

(a) Fuzzy semi-closed [1] if \( 1 - \lambda \) is fuzzy semi-open.

(b) Fuzzy pre-closed [2] if \( 1 - \lambda \) is fuzzy pre-open.
Definition 2.8 [7]. A function \( f : (X, T) \to (Y, S) \) from a fuzzy topological space \((X, T)\) into another fuzzy topological space \((Y, S)\) is called fuzzy irresolute if \( f^{-1}(\emptyset) \) is a fuzzy semi-open set in \((X, T)\) for any fuzzy semi-open set \( \emptyset \) in \((Y, S)\).

Definition 2.9 [11]. A function \( f : (X, T) \to (Y, S) \) from a fuzzy topological space \((X, T)\) into another fuzzy topological space \((Y, S)\) is called fuzzy pre-semi open if \( f^{-1}(\emptyset) \) is a fuzzy semi-open set in \((Y, S)\) for any fuzzy semi-open set \( \emptyset \) in \((X, T)\).

Definition 2.10 [2]. A function \( f : (X, T) \to (Y, S) \) from a fuzzy topological space \((X, T)\) into another fuzzy topological space \((Y, S)\) is called fuzzy pre-continuous if \( f^{-1}(\mu) \) is a fuzzy pre-open set in \((X, T)\) for any fuzzy open set \( \mu \) in \((Y, S)\).

Definition 2.11 [1]. A function \( f : (X, T) \to (Y, S) \) from a fuzzy topological space \((X, T)\) into another fuzzy topological space \((Y, S)\) is called fuzzy `semi-continuous if \( f^{-1}(\delta) \) is a fuzzy semi-open set in \((X, T)\) for any fuzzy open set \( \delta \) in \((Y, S)\).

Definition 2.12 [9]. A function \( f : (X, T) \to (Y, S) \) from a fuzzy topological space \((X, T)\) into another fuzzy topological space \((Y, S)\) is called somewhat fuzzy continuous if \( \lambda \in S \) and \( f^{-1}(\lambda) \neq \emptyset \) implies that there exist a fuzzy open set \( \delta \) in \((X, T)\) such that \( \delta \neq \emptyset \) and \( \delta \leq f^{-1}(\lambda) \). That is, \( \text{int} \left( f^{-1}(\lambda) \right) \neq \emptyset \).

Definition 2.13 [9]. A function \( f : (X, T) \to (Y, S) \) from a fuzzy topological space \((X, T)\) into another fuzzy topological space \((Y, S)\) is called somewhat fuzzy open if \( \lambda \in T \) and \( \lambda \neq 0 \) implies that there exists a fuzzy open set \( \eta \) in \((Y, S)\) such that \( \eta \neq 0 \) and \( \eta \leq f(\lambda) \). That is, \( \text{int} \left[ f(\lambda) \right] \neq \emptyset \).

3. Somewhat fuzzy nearly continuous functions

Motivated by the classical concept introduced in [8] we shall now define:

Definition 3.1. Let \((X, T)\) and \((Y, S)\) be any two fuzzy topological spaces. A function \( f : (X, T) \to (Y, S) \) is called a somewhat fuzzy nearly continuous function if \( \lambda \in S \) and \( f^{-1}(\lambda) \neq 0 \), there exists a non-zero fuzzy open set \( \mu \) of \((X, T)\) such that \( \mu \leq \text{cl} \left( f^{-1}(\lambda) \right) \). That is, \( f : (X, T) \to (Y, S) \) is a somewhat fuzzy nearly continuous function if \( \text{int} \text{cl} f^{-1}(\lambda) \neq 0 \), for any non-zero fuzzy open set \( \lambda \) in \((Y, S)\).
Example 3.1. Let $X = \{a, b, c\}$. The fuzzy sets $\lambda, \mu, \alpha$ and $\beta$ are defined on $X$ as follows:

$\lambda : X \rightarrow [0,1]$ is defined as $\lambda(a) = 0.7 ; \lambda(b) = 0.8 ; \lambda(c) = 0.6$.

$\mu : X \rightarrow [0,1]$ is defined as $\mu(a) = 0.9 ; \mu(b) = 0.6 ; \mu(c) = 0.5$.

$\alpha : X \rightarrow [0,1]$ is defined as $\alpha(a) = 0.8 ; \alpha(b) = 0.7 ; \alpha(c) = 0.6$.

$\beta : X \rightarrow [0,1]$ is defined as $\beta(a) = 0.6 ; \beta(b) = 0.9 ; \beta(c) = 0.5$.

Let $T = \{0, \lambda, \mu, \lambda \vee \mu, \lambda \wedge \mu, 1\}$ and $S = \{0, \alpha, \beta, \alpha \vee \beta, \alpha \wedge \beta, 1\}$. Then $T$ and $S$ are clearly fuzzy topologies on $X$. Define a function $f : (X, T) \rightarrow (Y, S)$ by $f(a) = b$; $f(b) = a$; $f(c) = c$. Then, for the non-zero fuzzy open sets $\alpha, \beta, [\alpha \vee \beta], [\alpha \wedge \beta]$ and 1 in $(X, S)$, we have $\text{int cl} [f^{-1}(\alpha)] \neq 0$, $\text{int cl} [f^{-1}(\beta)] \neq 0$, $\text{int cl} [f^{-1}(\alpha \vee \beta)] \neq 0$, $\text{int cl} [f^{-1}(\alpha \wedge \beta)] \neq 0$ and $\text{int cl} [f^{-1}(1)] \neq 0$ respectively in $(X, T)$. Hence $f$ is a somewhat fuzzy nearly continuous function from $(X, T)$ into $(X, S)$.

Example 3.2. Let $X = \{a, b, c\}$. The fuzzy sets $\lambda, \mu, \alpha$ and $\beta$ are defined on $X$ as follows:

$\lambda : X \rightarrow [0,1]$ is defined as $\lambda(a) = 0.4 ; \lambda(b) = 0.6 ; \lambda(c) = 0.4$.

$\mu : X \rightarrow [0,1]$ is defined as $\mu(a) = 0.6 ; \mu(b) = 0.5 ; \mu(c) = 0.4$.

$\alpha : X \rightarrow [0,1]$ is defined as $\alpha(a) = 0.3 ; \alpha(b) = 0.5 ; \alpha(c) = 0.2$.

$\beta : X \rightarrow [0,1]$ is defined as $\beta(a) = 0.5 ; \beta(b) = 0.4 ; \beta(c) = 0.3$.

Let $T = \{0, \lambda, \mu, \lambda \vee \mu, \lambda \wedge \mu, 1\}$ and $S = \{0, \alpha, \beta, \alpha \vee \beta, \alpha \wedge \beta, 1\}$. Then $T$ and $S$ are clearly fuzzy topologies on $X$. Define a function $f : (X, T) \rightarrow (Y, S)$ by $f(a) = a$; $f(b) = b$; $f(c) = c$. Then, for the non-zero fuzzy open sets $\alpha, \beta, [\alpha \vee \beta], [\alpha \wedge \beta]$ and 1 in $(X, S)$, we have $\text{int cl} [f^{-1}(\alpha)] = 1 - [\lambda \wedge \mu] \neq 0$, $\text{int cl} [f^{-1}(\beta)] = 1 - [\lambda \vee \mu] \neq 0$, $\text{int cl} [f^{-1}(\alpha \vee \beta)] = 0$, $\text{int cl} [f^{-1}(\alpha \wedge \beta)] = \mu \neq 0$ and $\text{int cl} [f^{-1}(1)] \neq 0$ respectively in $(X, T)$. Since $\text{int cl} [f^{-1}(\alpha \vee \beta)] = 0$, $f$ is not a somewhat fuzzy nearly continuous function from $(X, T)$ into $(X, S)$.

Proposition 3.1. If a function $f : (X, T) \rightarrow (Y, S)$ from a fuzzy topological space $(X, T)$ into another fuzzy topological space $(Y, S)$ is a somewhat fuzzy continuous function, then $f$ is a somewhat fuzzy nearly continuous function.

Proof. Let $\lambda$ be a non-zero fuzzy open set in $(Y, S)$ such that $f^{-1}(\lambda) \neq 0$. Since $f$ is a somewhat fuzzy continuous function, there exists a non-zero fuzzy open set $\mu$ of $(X, T)$ such that $\mu \leq f^{-1}(\lambda)$. Now $f^{-1}(\lambda) \subseteq \text{cl} [f^{-1}(\lambda)]$, implies that $\mu \subseteq \text{cl} [f^{-1}(\lambda)]$. That is, $\text{int cl} [f^{-1}(\lambda)] \neq 0$. Hence $f$ is a somewhat fuzzy nearly continuous function.
Remarks. The implications contained in the following diagram are true.

\[ \text{Fuzzy continuity} \rightarrow \text{Somewhat fuzzy continuity} \rightarrow \text{Fuzzy pre-continuity} \rightarrow \text{Somewhat fuzzy nearly continuity} \]

However, none of the above implications is reversed as is shown in the following examples:

**Example 3.3.** Let \( X = \{a,b,c\} \). The fuzzy sets \( \lambda \), \( \mu \), \( \alpha \) and \( \beta \) are defined on \( X \) as follows:
- \( \lambda : X \rightarrow [0,1] \) is defined as \( \lambda(a) = 1 \); \( \lambda(b) = 0.2 \); \( \lambda(c) = 0.7 \).
- \( \mu : X \rightarrow [0,1] \) is defined as \( \mu(a) = 0.3 \); \( \mu(b) = 1 \); \( \mu(c) = 0.2 \).
- \( \vartheta : X \rightarrow [0,1] \) is defined as \( \vartheta(a) = 0.7 \); \( \vartheta(b) = 0.4 \); \( \vartheta(c) = 1 \).
- \( \alpha : X \rightarrow [0,1] \) is defined as \( \alpha(a) = 0.6 \); \( \alpha(b) = 0.5 \); \( \alpha(c) = 0.7 \).
- \( \beta : X \rightarrow [0,1] \) is defined as \( \beta(a) = 0.7 \); \( \beta(b) = 0.3 \); \( \beta(c) = 0.5 \).

Let \( T = \{0,\lambda,\mu,\lambda \lor \vartheta,\mu \lor \vartheta,\lambda \land \mu,\lambda \land \vartheta,\mu \lor \lambda \land \vartheta,\lambda \lor \mu\} \) and \( S = \{0,\alpha,\beta,\alpha \lor \beta,\alpha \land \beta,\lambda \land \mu\} \). Then \( T \) and \( S \) are clearly fuzzy topologies on \( X \). Define a function \( f : (X,T) \rightarrow (X,S) \) by \( f(a) = a \); \( f(b) = b \); \( f(c) = c \). Then, for the non-zero fuzzy open sets \( \alpha \), \( \beta \), \( [\alpha \lor \beta] \), \( [\alpha \land \beta] \) and 1 in \( (X,S) \), we have \( \text{int cl}[f^{-1}(\alpha)] = \text{int cl}[f^{-1}(\beta)] = \text{int cl}[f^{-1}(\alpha \lor \beta)] = \text{int cl}[f^{-1}(\alpha \land \beta)] = \vartheta \land [\lambda \lor \mu] \neq 0 \) and \( \text{int cl}[f^{-1}(1)] \neq 0 \) respectively in \( (X,T) \). Hence \( f \) is a somewhat fuzzy nearly continuous function from \( (X,T) \) into \( (X,S) \). But \( f \) is not a fuzzy pre-continuous function from \( (X,T) \) into \( (X,S) \), since in \( (X,T) \), we have \( (\alpha \lor \beta) \notin \text{int cl}[f^{-1}(\alpha \lor \beta)] = \vartheta \land [\lambda \lor \mu] \). Also \( f \) is not a fuzzy continuous function from \( (X,T) \) into \( (X,S) \), since \( [f^{-1}(\alpha)] = \alpha \notin T \).

**Example 3.4.** Let \( X = \{a,b,c\} \). The fuzzy sets \( \lambda \), \( \mu \), \( \alpha \) and \( \beta \) are defined on \( X \) as follows:
- \( \lambda : X \rightarrow [0,1] \) is defined as \( \lambda(a) = 0.8 \); \( \lambda(b) = 0.6 \); \( \lambda(c) = 0.7 \).
- \( \mu : X \rightarrow [0,1] \) is defined as \( \mu(a) = 0.6 \); \( \mu(b) = 0.9 \); \( \mu(c) = 0.8 \).
- \( \vartheta : X \rightarrow [0,1] \) is defined as \( \vartheta(a) = 0.7 \); \( \vartheta(b) = 0.5 \); \( \vartheta(c) = 0.9 \).
- \( \alpha : X \rightarrow [0,1] \) is defined as \( \alpha(a) = 0.7 \); \( \alpha(b) = 0.8 \); \( \alpha(c) = 0.6 \).
- \( \beta : X \rightarrow [0,1] \) is defined as \( \beta(a) = 0.9 \); \( \beta(b) = 0.6 \); \( \beta(c) = 0.5 \).
\[ \delta : X \to [0,1] \] is defined as \( \delta(a) = 0.6; \delta(b) = 0.9; \delta(c) = 0.7. \)

Let \( T = \{0, \lambda, \mu, \varnothing, \lambda \lor \mu, \lambda \land \varnothing, \lambda \lor \varnothing, \lambda \land \varnothing, \lambda \lor \mu \land \varnothing, \lambda \land \mu \lor \varnothing, \lambda \lor \mu \land \varnothing, \lambda \lor \mu \land \varnothing, \lambda \lor \mu \land \varnothing, 1\} \) and \( S = \{0, \alpha, \beta, \delta, \alpha \lor \beta, \alpha \land \delta, \beta \lor \delta, \beta \land \delta, \} \). Then \( T \) and \( S \) are clearly fuzzy topologies on \( X \). Define a function \( f : (X, T) \to (X, S) \) by \( f(a) = a; f(b) = b; f(c) = c. \)

Then, \( f \) is a somewhat fuzzy nearly continuous function from \((X, T)\) into \((X, S)\). But \( f \) is not a somewhat fuzzy continuous function from \((X, T)\) into \((X, S)\). Since \( \text{int}[f^{-1}(\alpha)] = \text{int}(\alpha) = 0, f \) is not a fuzzy continuous function.

**Proposition 3.2.** Let \((X, T)\) and \((Y, S)\) be any two fuzzy topological spaces. Let \( f : (X, T) \to (Y, S) \) be a non-to one and onto function. Then the following are equivalent:

1. \( f \) is somewhat fuzzy nearly continuous.
2. If \( \lambda \) is a fuzzy open and fuzzy dense set in \((X, T)\), then \( f(\lambda) \) is a fuzzy dense set in \((Y, S)\).
3. If \( \lambda \) is a fuzzy nowhere dense in \((X, T)\), then \( \text{int}[f(\lambda)] = 0 \) in \((Y, S)\).

**Proof.** (1) \( \Rightarrow \) (2) Let \( f \) be a somewhat fuzzy nearly continuous function from a fuzzy topological space \((X, T)\) into a fuzzy topological space \((Y, S)\) and suppose that \( \lambda \) is a fuzzy open and fuzzy dense set in \((X, T)\). We claim that \( \text{cl}[f(\lambda)] = 1. \) Assume the contrary. Then \( \text{cl}[f(\lambda)] \neq 1. \) Then there exists a non-zero closed set \( \delta \) of \((Y, S)\) such that \( f(\lambda) < \delta < 1. \) Then \( f^{-1}(\delta) \) is not a fuzzy continuous function from \((X, T)\) into \((Y, S)\). Since \( \delta \neq 1 \) is a non-zero fuzzy closed set \( \delta \) of \((Y, S)\), \( 1-\delta \) is a non-zero fuzzy open set in \((Y, S)\). By hypothesis, \( \text{int}[f^{-1}(1-\delta)] \neq 0. \) Then \( \text{int}[1-f^{-1}(\delta)] \neq 0. \) That is, \( \text{cl}[1-f^{-1}(\delta)] = 1. \) Now \( \lambda < f^{-1}(\delta), \) implies that \( \lambda \lor \mu \land \varnothing \lor f^{-1}(\delta), \) implies that \( \lambda \lor \mu \land \varnothing \lor f^{-1}(\delta) \) is a fuzzy open set in \((X, T)\). Hence we must have \( \text{cl}[f(\lambda)] = 1. \)

(2) \( \Rightarrow \) (3) Suppose that \( \lambda \) is a fuzzy nowhere dense in \((X, T)\). Then we have \( \text{int}[\lambda] = 0. \) Now \( 1-\text{int}[\lambda] = 1-0 = 1, \) implies that \( \text{cl}[1-\lambda] = 1. \) Hence \( 1-\lambda \) is a fuzzy open and fuzzy dense set in \((X, T)\). By hypothesis, \( f(\text{int}[1-\lambda]) \) is a fuzzy dense set in \((Y, S)\). That is, \( \text{cl}[f(\text{int}[1-\lambda])] = 1. \) Now \( \text{int}[1-\lambda] \leq 1-\lambda \) implies that \( \text{cl}[f(\text{int}[1-\lambda])] \leq \text{cl}[f(1-\lambda)]. \) Then we have \( 1 \leq \text{cl}[f(1-\lambda)]. \) That is, \( \text{cl}[f(1-\lambda)] = 1. \)
Since the function $f$ is one to one and onto, $f\left(\left[1-\lambda\right]\right)=1-f(\lambda)$. Then $\text{cl}\left[1-f(\lambda)\right]=1$.

This implies that $1-\text{int}[f(\lambda)]=1$. Therefore $\text{int}\left[f(\lambda)\right]=0$ in $(Y, S)$.

(3) $\Rightarrow$ (1). Suppose that $\lambda$ is a fuzzy open in $(Y, S)$. We claim that $\text{int} f^{-1}(\lambda) \neq 0$.

Assume the contrary. That is, $\text{int} f^{-1}(\lambda)=0$. This implies that $f^{-1}(\lambda)$ is a fuzzy nowhere dense in $(X, T)$. Then, by hypothesis, $\text{int}\left[f\left(f^{-1}(\lambda)\right)\right]=0$ in $(Y, S)$. Since the function $f$ is one to one and onto, $f\left(f^{-1}(\lambda)\right)=\lambda$. Then we have $\text{int}(\lambda)=0$, which implies that $\lambda=0$ (since $\lambda$ is fuzzy open in $(Y, S)$, $\text{int}(\lambda)=\lambda$). But this is a contradiction to $\lambda$ being a non-zero fuzzy (open) set in $(Y, S)$. Hence we must have $\text{int} f^{-1}(\lambda) \neq 0$.

**Theorem 3.1** [10]. If a non-zero fuzzy set $\lambda$ in a fuzzy topological space $(X, T)$, is a fuzzy nowhere dense set, then $\lambda$ is a fuzzy semi-closed set in $(X, T)$.

**Proposition 3.3.** Let $f:(X, T) \rightarrow (Y, S)$ be a one-to-one, fuzzy pre semi-open function from a fuzzy topological space $(X, T)$ onto the fuzzy topological space $(Y, S)$. Then the following are equivalent:

1. $f$ is somewhat fuzzy nearly continuous.
2. If $\lambda$ is a fuzzy nowhere dense set in $(X, T)$, then $f(\lambda)$ is a fuzzy nowhere dense set in $(Y, S)$.

**Proof.** (1) $\Rightarrow$ (2) Assume (1) suppose that $\lambda$ is a fuzzy nowhere dense set in $(X, T)$. Then by Theorem 3.1, $\lambda$ is a fuzzy semi-closed set in $(X, T)$. Hence $1-\lambda$ is a fuzzy semi-open set in $(X, T)$. Since the function $f$ is a fuzzy pre semi-open function, $f(1-\lambda)$ is a fuzzy semi-open set in $(Y, S)$. Then we have $f(1-\lambda) \leq \text{cl}\text{int}[f(1-\lambda)]$. Since $f$ is one-to-one and onto, $f(1-\lambda)=1-f(\lambda)$. Then we have $1-f(\lambda) \leq \text{cl}\text{int}[1-f(\lambda)]$. This implies that $1-f(\lambda) \leq \text{int}\text{cl}[f(\lambda)]$. Then we have $\text{int}\text{cl}[f(\lambda)] \leq f(\lambda)$. Now $\text{int}\text{cl}[f(\lambda)] \leq \text{int}[f(\lambda)]$ implies that $\text{int}[f(\lambda)] \leq \text{int}[f(\lambda)]$ (since $\text{cl}[f(\lambda)]$ is fuzzy open in $(Y, S)$, $\text{int}[\text{cl}[f(\lambda)]]=\text{int}[f(\lambda)$]. Then, we have $\text{int}\text{cl}[f(\lambda)] \leq \text{int}[f(\lambda)]$. Since $f$ is a somewhat fuzzy nearly continuous function and $\lambda$ is a fuzzy nowhere dense set in $(X, T)$, by Proposition 3.2, $\text{int}[f(\lambda)]=0$ in $(Y, S)$. Hence from (A), we have, $\text{int}\text{cl}[f(\lambda)] \leq 0$. That is, $\text{int}\text{cl}[f(\lambda)]=0$ in $(Y, S)$. Therefore $f(\lambda)$ is a fuzzy nowhere dense set in $(Y, S)$.

(2) $\Rightarrow$ (1) Assume (2). Suppose that $\lambda$ is a fuzzy open and fuzzy dense set in $(X, T)$. Then $\text{int}(\lambda)=\lambda$ and $\text{cl}(\lambda)=1$, implies that $\text{cl}\text{int}(\lambda)=1$. Then we have $1-\text{cl}\text{int}(\lambda)=0$. 


This implies that \( \text{int cl}(1-\lambda) = 0 \). Hence \( 1-\lambda \) is a fuzzy nowhere dense set in \((X,T)\).

By hypothesis, \( f(1-\lambda) \) is a fuzzy nowhere dense set in \((Y,S)\). Then, \( \text{int cl}\left[(1-f(\lambda))\right] = 0 \). Since \( f \) is one-to-one and onto, \( \text{int cl}(1-f(\lambda)) = 0 \). This implies that \( 1-\text{cl int}(f(\lambda)) = 0 \). Since \( \text{int}(f(\lambda)) \leq (f(\lambda)) \), \( 1-\text{cl int}(f(\lambda)) \leq 1- \text{cl int}(f(\lambda)) \). Therefore, we have

\[
1-\text{cl int}\left[(f(\lambda))\right] \leq 0.
\]

That is, \( 1-\text{cl int}(f(\lambda)) = 0 \) implies that \( \text{cl int}(f(\lambda)) = 1 \). Hence \( f(\lambda) \) is a fuzzy dense set in \((Y,S)\). Hence, for the fuzzy open and fuzzy dense set \( \lambda \) in \((X,T)\), we have \( \text{cl int}(f(\lambda)) = 1 \). Therefore, by Proposition 3.2, \( f \) is a somewhat fuzzy nearly continuous function.

**Proposition 3.4.** Let \( f:(X,T) \to (Y,S) \) be a one-to-one, fuzzy pre semi-open and somewhat fuzzy nearly continuous function from a fuzzy topological space \((X,T)\) onto the fuzzy topological space \((Y,S)\). If \( \lambda \) is a fuzzy first category in \((X,T)\), then \( f(\lambda) \) is a fuzzy first category in \((Y,S)\).

**Proof.** Let \( \lambda \) be a fuzzy first category in \((X,T)\). Then \( \lambda = \bigvee_{\alpha=1} \lambda_{\alpha} \), where \( \lambda_{\alpha} \)’s are fuzzy nowhere dense sets in \((X,T)\). Now \( f(\lambda) = f\left(\bigvee_{\alpha=1} \lambda_{\alpha}\right) = \bigvee_{\alpha=1} f(\lambda_{\alpha}) \). Since \( f \) is a one-to-one, fuzzy pre semi-open and somewhat fuzzy nearly continuous function from \((X,T)\) onto \((Y,S)\) and \( \lambda_{\alpha} \)’s are fuzzy nowhere dense sets in \((X,T)\), by Proposition 3.3, \( f(\lambda_{\alpha}) \)’s are fuzzy nowhere dense sets in \((Y,S)\). Hence \( f(\lambda) = \bigvee_{\alpha=1} f(\lambda_{\alpha}) \), where \( f(\lambda_{\alpha}) \)’s are fuzzy nowhere dense sets in \((Y,S)\), implies that \( f(\lambda) \) is a fuzzy first category in \((Y,S)\).

**Proposition 3.5.** If \( f:(X,T) \to (Y,S) \) is a somewhat fuzzy nearly continuous function from a fuzzy topological space \((X,T)\) into a fuzzy topological space \((Y,S)\) and \( g:(Y,S) \to (Z,W) \) is a fuzzy continuous function from \((Y,S)\) into a fuzzy topological space \((Z,W)\), then \( g \circ f:(X,T) \to (Z,W) \) is a somewhat fuzzy nearly continuous function from \((X,T)\) into \((Z,W)\).

**Proof.** Let \( \lambda \) be a non-zero fuzzy open set in \((Z,W)\). Since \( g \) is a fuzzy continuous function from \((Y,S)\) into \((Z,W)\), \( g^{-1}(\lambda) \) is a fuzzy open set in \((Y,S)\). Since \( f \) is a somewhat fuzzy nearly continuous function from \((X,T)\) into \((Y,S)\), there exists a non-zero fuzzy open set \( \mu \) of \((X,T)\) such that \( \mu \preceq \text{cl f}^{-1}(g^{-1}(\lambda)) \). That is, \( \mu \preceq \text{cl f}(g \circ f)^{-1}(\lambda) \). Hence \( \text{int cl}(g \circ f)^{-1}(\lambda) \neq 0 \). Therefore \( g \circ f \) is a somewhat fuzzy nearly continuous function from \((X,T)\) into \((Z,W)\).
Proposition 3.6. If \( f : (X, T) \to (Y, S) \) is a function from a fuzzy topological space \((X, T)\) into a fuzzy topological space \((Y, S)\) and the graph function \( g : X \to X \times Y \) of \( f \) is somewhat fuzzy nearly continuous, then \( f \) is somewhat fuzzy nearly continuous.

Proof. Let \( \lambda \) be a non-zero fuzzy open set in \((Y, S)\). Then \( 1 \times \lambda \) is a fuzzy open set in \( X \times Y \). Since \( g \) is somewhat fuzzy nearly continuous, we have \( \text{int}[g^{-1}(1 \times \lambda)] \neq \emptyset \). But \( f^{-1}(\lambda) = 1 \land f^{-1}(\lambda) = g^{-1}(1 \times \lambda) \). This implies that \( \text{int}[f^{-1}(\lambda)] \neq \emptyset \). Hence \( f \) is a somewhat fuzzy nearly continuous function from \((X, T)\) into \((Y, S)\).

4. Somewhat fuzzy nearly open functions

Motivated by the classical concept introduced in [5] we shall now define:

Definition 4.1. Let \((X, T)\) and \((Y, S)\) be any two fuzzy topological spaces. A function \( f : (X, T) \to (Y, S) \) is called a somewhat fuzzy nearly open function if for all \( T \in T \) and \( f^{-1}(\lambda) \neq \emptyset \), there exists a non-zero fuzzy open set \( \mu \) of \((Y, S)\) such that \( \text{cl}[f^{-1}(\lambda)] \neq \emptyset \). That is, a function \( f : (X, T) \to (Y, S) \) is a somewhat fuzzy nearly open function if \( \text{int}[\text{cl} f^{-1}(\lambda)] \neq \emptyset \), for every non-zero fuzzy open set \( \lambda \) of \((X, T)\).

Example 4.1. Let \( X = \{a, b, c\} \). The fuzzy sets \( \lambda, \mu, \alpha \) and \( \beta \) are defined on \( X \) as follows:

\[
\begin{align*}
\lambda : X & \to [0,1] \text{ defined as } \lambda(a) = 0.4; \lambda(b) = 0.6; \lambda(c) = 0.4. \\
\mu : X & \to [0,1] \text{ defined as } \mu(a) = 0.6; \mu(b) = 0.5; \mu(c) = 0.4. \\
\alpha : X & \to [0,1] \text{ defined as } \alpha(a) = 0.3; \alpha(b) = 0.5; \alpha(c) = 0.2. \\
\beta : X & \to [0,1] \text{ defined as } \beta(a) = 0.5; \beta(b) = 0.4; \beta(c) = 0.3.
\end{align*}
\]

Let \( T = \{0, \lambda, \mu, \lambda \lor \mu, \lambda \land \mu, 1\} \) and \( S = \{0, \alpha, \beta, \alpha \lor \beta, \alpha \land \beta, 1\} \). Then \( T \) and \( S \) are clearly fuzzy topologies on \( X \). Define a function \( f : (X, T) \to (X, S) \) by \( f(a) = a \); \( f(b) = b \); \( f(c) = c \). Then, for the non-zero fuzzy open sets \( \lambda, \mu, \lambda \lor \mu, \lambda \land \mu \), and \( 1 \) in \((X, T)\), we have \( \text{int}[f^-1(\lambda)] = \text{int}(1 - \beta) = \lambda \neq 0 \), \( \text{int}[f^-1(\mu)] = \text{int}(1 - \alpha) = \mu \neq 0 \), \( \text{int}[f^-1(\lambda \lor \mu)] = \text{int}(1 - [\alpha \lor \beta]) = \lambda \lor \mu \neq 0 \), \( \text{int}[f^-1(\lambda \land \mu)] = \text{int}(1 - [\alpha \land \beta]) = \lambda \land \mu \neq 0 \) and \( \text{int}[f^-1(1)] \neq 0 \) respectively in \((X, S)\). Hence \( f \) is a somewhat fuzzy nearly open function from \((X, T)\) into \((X, S)\).

Example 4.2. Let \( X = \{a, b\} \). The fuzzy sets \( \lambda, \mu, \alpha \) and \( \beta \) are defined on \( X \) as follows:

\[
\begin{align*}
\lambda : X & \to [0,1] \text{ defined as } \lambda(a) = 0.2; \lambda(b) = 0.4.
\end{align*}
\]
\( \mu : X \to [0,1] \) is defined as \( \mu(a) = 0.5 \); \( \mu(b) = 0.3 \).

\( \alpha : X \to [0,1] \) is defined as \( \alpha(a) = 0.5 \); \( \alpha(b) = 0.7 \).

\( \beta : X \to [0,1] \) is defined as \( \beta(a) = 0.8 \); \( \beta(b) = 0.4 \).

Let \( T = \{0, \lambda, \mu, \lambda \vee \mu, \lambda \wedge \mu, 1\} \) and \( S = \{0, \alpha, \beta, \alpha \vee \beta, \alpha \wedge \beta, 1\} \). Then \( T \) and \( S \) are clearly fuzzy topologies on \( X \). Define a function \( f : (X,T) \to (X,S) \) by \( f(a) = a \); \( f(b) = b \). Then, for the non-zero fuzzy open sets \( \lambda, \mu \) in \((X,T)\), we have \( \text{int} \left[ f(\lambda) \right] = \text{int}(1-\beta) = 0 \) and \( \text{int} \left[ f(\mu) \right] = \text{int}(1-\alpha) = 0 \) in \((X,S)\). Hence \( f \) is not a somewhat fuzzy nearly open function from \((X,T)\) into \((X,S)\).

**Proposition 4.1.** If a function \( f : (X,T) \to (Y,S) \) from a fuzzy topological space \((X,T)\) into another fuzzy topological space \((Y,S)\) is a somewhat fuzzy open function, then \( f \) is a somewhat fuzzy nearly open function.

**Proof.** Let \( \lambda \) be a non-zero fuzzy open set in \((X,T)\) such that \( f(\lambda) \neq 0 \). Since \( f \) is a somewhat fuzzy open function, there exists a non-zero fuzzy open set \( \mu \) of \((X,T)\) such that \( \mu \leq f(\lambda) \). Now \( f(\lambda) \leq \text{cl} \left[ f(\lambda) \right] \), implies that \( \mu \leq \text{cl} \left[ f(\lambda) \right] \). That is, \( \text{int} \left[ f(\lambda) \right] \neq 0 \). Hence \( f \) is a somewhat fuzzy nearly open function.

**Remarks.** The implications contained in the following diagram are true.

![Diagram](image)

However, none of the above implications is reversed as is shown in the following examples:

**Example 4.3.** Let \( X = \{a,b,c\} \). The fuzzy sets \( \lambda \), \( \mu \), \( \alpha \) and \( \beta \) are defined on \( X \) as follows:

\( \lambda : X \to [0,1] \) is defined as \( \lambda(a) = 0.4 \); \( \lambda(b) = 0.6 \); \( \lambda(c) = 0.4 \).

\( \mu : X \to [0,1] \) is defined as \( \mu(a) = 0.6 \); \( \mu(b) = 0.5 \); \( \mu(c) = 0.4 \).

\( \alpha : X \to [0,1] \) is defined as \( \alpha(a) = 0.6 \); \( \alpha(b) = 0.5 \); \( \alpha(c) = 0.8 \).

\( \beta : X \to [0,1] \) is defined as \( \beta(a) = 0.4 \); \( \beta(b) = 0.9 \); \( \beta(c) = 0.7 \).

Let \( T = \{0, \lambda, \mu, \lambda \vee \mu, \lambda \wedge \mu, 1\} \) and \( S = \{0, \alpha, \beta, \alpha \vee \beta, \alpha \wedge \beta, 1\} \). Then \( T \) and \( S \) are clearly fuzzy topologies on \( X \). Define a function \( f : (X,T) \to (X,S) \) by \( f(a) = a \);
Let \( f(b) = b; f(c) = c \). Then, for the non-zero fuzzy sets \( \lambda, \mu, \lambda \vee \mu, \lambda \wedge \mu \) and 1 in \((X,T)\), we have \( \text{int} \left[ f(\lambda) \right] \neq 0, \text{int} \left[ f(\mu) \right] \neq 0, \text{int} \left[ f(\lambda \vee \mu) \right] \neq 0, \text{int} \left[ f(\lambda \wedge \mu) \right] \neq 0 \) and \( \text{int} \left[ f(1) \right] \neq 0 \) respectively in \((X,S)\). Hence \( f \) is a somewhat fuzzy nearly open function from \((X,T)\) into \((X,S)\). But \( f \) is not a somewhat fuzzy open function, since \( \text{int} \left[ f(\lambda) \right] = 0 \) in \((X,S)\).

Also \( f \) is not a fuzzy open function, since \( f(\lambda) \notin S \) in \((X,S)\).

**Example 4.4.** Let \( X = \{a, b, c\} \). The fuzzy sets \( \alpha, \beta, \theta, \lambda, \mu \) and \( \delta \) are defined on \( X \) as follows:

\[
\alpha: X \to [0,1] \text{ is defined as } \alpha(a) = 0.6; \alpha(b) = 0.5; \alpha(c) = 0.4. \\
\beta: X \to [0,1] \text{ is defined as } \beta(a) = 0.5; \beta(b) = 0.6; \beta(c) = 0.7. \\
\theta: X \to [0,1] \text{ is defined as } \theta(a) = 0.4; \theta(b) = 0.7; \theta(c) = 0.5. \\
\lambda: X \to [0,1] \text{ is defined as } \lambda(a) = 0.4; \lambda(b) = 0.6; \lambda(c) = 0.5. \\
\mu: X \to [0,1] \text{ is defined as } \mu(a) = 0.7; \mu(b) = 0.5; \mu(c) = 0.4. \\
\delta: X \to [0,1] \text{ is defined as } \delta(a) = 0.3; \delta(b) = 0.4; \delta(c) = 0.6.
\]

Let \( T = \{0, \alpha, \beta, \theta, \alpha \vee \beta, \alpha \wedge \beta, \alpha \wedge \theta, \beta \wedge \alpha, \beta \wedge \theta, \alpha \wedge \beta, \alpha \vee \beta \wedge \theta, \alpha \vee \beta \wedge \delta, \alpha \wedge \beta \vee \theta, \alpha \wedge \beta \wedge \delta, \beta \wedge \alpha \vee \theta, \beta \wedge \alpha \wedge \delta, \beta \wedge \theta \vee \alpha, \beta \wedge \theta \wedge \delta, \mu \vee \delta, \mu \vee \alpha \wedge \beta, \mu \vee \beta \vee \alpha \wedge \delta, \mu \vee \beta \wedge \alpha \vee \theta, \mu \vee \beta \wedge \alpha \wedge \delta, \mu \vee \beta \wedge \theta \vee \alpha, \mu \vee \beta \wedge \theta \wedge \delta, \mu \wedge \delta \wedge \alpha, \mu \wedge \delta \wedge \beta, \mu \wedge \delta \wedge \theta, \mu \wedge \delta \vee \alpha, \mu \wedge \delta \vee \beta, \mu \wedge \delta \vee \theta, \mu \wedge \delta \vee \alpha \wedge \beta, \mu \wedge \delta \vee \beta \vee \alpha \wedge \delta, \mu \wedge \delta \vee \beta \wedge \alpha \vee \theta, \mu \wedge \delta \vee \beta \wedge \alpha \wedge \delta, \mu \wedge \delta \vee \beta \wedge \theta \vee \alpha, \mu \wedge \delta \vee \beta \wedge \theta \wedge \delta, \mu \wedge \delta \vee \alpha \vee \beta, \mu \wedge \delta \vee \alpha \wedge \beta, \mu \wedge \delta \vee \beta \wedge \alpha \vee \theta, \mu \wedge \delta \vee \beta \wedge \alpha \wedge \delta, \mu \wedge \delta \vee \beta \wedge \theta \vee \alpha, \mu \wedge \delta \vee \beta \wedge \theta \wedge \delta\} \) and \( S = \{0, \lambda, \mu, \delta \vee \lambda, \delta \wedge \lambda, \lambda \vee \mu, \delta \wedge \mu, \lambda \vee \delta, \delta \wedge \alpha, \lambda \wedge \alpha, \lambda \wedge \delta, \lambda \wedge \beta, \mu \vee \delta, \lambda \wedge \mu \vee \delta, \lambda \wedge \mu \vee \alpha \wedge \delta, \lambda \wedge \mu \vee \beta \wedge \delta, \lambda \wedge \mu \vee \alpha \wedge \beta, \lambda \wedge \mu \vee \alpha \wedge \theta, \lambda \wedge \mu \vee \alpha \wedge \delta, \lambda \wedge \mu \vee \alpha \wedge \beta, \lambda \wedge \mu \vee \beta \wedge \delta, \lambda \wedge \mu \vee \beta \wedge \theta \wedge \delta, \lambda \wedge \mu \vee \alpha \wedge \beta, \lambda \wedge \mu \vee \alpha \wedge \delta, \lambda \wedge \mu \vee \alpha \wedge \theta, \lambda \wedge \mu \vee \alpha \wedge \beta, \lambda \wedge \mu \vee \beta \wedge \delta, \lambda \wedge \mu \vee \beta \wedge \delta \} \). Then \( T \) and \( S \) are clearly fuzzy topologies on \( X \). Define a function \( f : (X,T) \to (X,S) \) by \( f(a) = a ; f(b) = c ; f(c) = b \). Now, for the non-zero fuzzy open set \( \alpha \) in \((X,T)\), \( \text{int} \left[ f(\alpha) \right] = \text{int} \left[ \alpha \wedge \delta \right] = \alpha \wedge \delta \) and \( f(\alpha) \subseteq \text{int} \left[ f(\alpha) \right] \). Then \( f(\alpha) \) is not a fuzzy pre-open set in \((X,S)\) and hence \( f \) is not a fuzzy pre-open function from \((X,T)\) into \((X,S)\) and \( f \) is a somewhat fuzzy nearly open function from \((X,T)\) into \((X,S)\), since \( \text{int} \left[ f(\eta) \right] \neq 0 \), for every non-zero fuzzy open set \( \eta \) of \((X,T)\).

**Proposition 4.2.** If \( f : (X,T) \to (Y,S) \) is a fuzzy open function from a fuzzy topological space \((X,T)\) into a fuzzy topological space \((Y,S)\) and \( g : (Y,S) \to (Z,W) \) is a somewhat fuzzy nearly open function from \((Y,S)\) into a fuzzy topological space \((Z,W)\), then \( g \circ f : (X,T) \to (Z,W) \) is a somewhat fuzzy nearly open function from \((X,T)\) into \((Z,W)\).

**Proof.** Let \( \lambda \) be a non-zero fuzzy open set in \((X,T)\). Since \( f \) is a fuzzy open function from \((X,T)\) into \((Y,S)\), \( f(\lambda) \) is a fuzzy open set in \((Y,S)\). Since \( g \) is a somewhat fuzzy nearly open function from \((Y,S)\) into \((Z,W)\), there exists a non-zero fuzzy open set \( \mu \) of \((Z,W)\) such that \( \mu \subseteq \text{cl} g(f(\lambda)) \). Since \( (g \circ f)(\lambda) = g(f(\lambda)) \), we
have \( \mu \leq \text{cl}(g \circ f)(\lambda) \). Hence \( \text{int} \text{cl}[(g \circ f)(\lambda)] \neq 0 \). Therefore \( g \circ f \) is a somewhat fuzzy nearly open function from \((X,T)\) into \((Z,W)\).

**Proposition 4.3.** Let \( f : (X,T) \to (Y,S) \) be a function from a fuzzy topological space \((X,T)\) onto a fuzzy topological space \((Y,S)\) and \( g : (Y,S) \to (Z,W) \) be a function from \((Y,S)\) into a fuzzy topological space \((Z,W)\). If \( f \) is a fuzzy continuous function and \( g \circ f \) is a somewhat fuzzy nearly open function from \((X,T)\) into \((Z,W)\), then \( g \) is a somewhat fuzzy nearly open function from \((Y,S)\) into \((Z,W)\).

**Proof.** Let \( \lambda \) be a non-zero fuzzy open set in \((Y,S)\). Since \( f \) is a fuzzy continuous function from \((X,T)\) into \((Y,S)\), \( f^{-1}(\lambda) \) is a fuzzy open set in \((X,T)\). Since \( g \circ f \) is a somewhat fuzzy nearly open function, \( \text{cl}(g \circ f)(f^{-1}(\lambda)) \neq 0 \). Then, \( \text{int} \text{cl}[(g \circ f)(f^{-1}(\lambda))] \neq 0 \). Since \( f \) is onto, \( f^{-1}(\lambda) = \lambda \). Then we have \( \text{int} \text{cl}(g(\lambda)) \neq 0 \). Hence \( g \) is a somewhat fuzzy nearly open function from \((Y,S)\) into \((Z,W)\).

**Proposition 4.4.** Let \((X,T)\) and \((Y,S)\) be any two fuzzy topological spaces. Let \( f : (X,T) \to (Y,S) \) be a function. Then the following conditions are equivalent:

1. \( f \) is somewhat fuzzy nearly open.

2. If \( \lambda \) is a fuzzy open and fuzzy dense set in \((Y,S)\), then \( f^{-1}(\lambda) \) is a fuzzy dense set in \((X,T)\).

3. If \( \lambda \) is a fuzzy nowhere dense in \((Y,S)\), then \( \text{int} f^{-1}(\lambda) = 0 \) in \((X,T)\).

**Proof.** (1) \( \Rightarrow \) (2) Assume (1). Suppose that \( \lambda \) is a fuzzy open and fuzzy dense set in \((Y,S)\). We claim that \( f^{-1}(\lambda) \) is a fuzzy dense set in \((X,T)\). Assume the contrary. Then \( \text{cl}[f^{-1}(\lambda)] \neq 1 \). Then \( 1 - \text{cl}[f^{-1}(\lambda)] \neq 0 \). This implies that \( \text{int}(1 - [f^{-1}(\lambda)]) \neq 0 \). Then \( \text{int}(f^{-1}(1 - \lambda)) \neq 0 \). Since \( f \) is a somewhat fuzzy nearly open function and \( \text{int}(f^{-1}(1 - \lambda)) \) is a fuzzy open set in \((X,T)\), \( \text{int} \text{cl}[(f^{-1}(1 - \lambda))] \neq 0 \). Now \( \text{int}(f^{-1}(1 - \lambda)) \leq f^{-1}(1 - \lambda) \) implies that \( \text{int} \text{cl}[(f^{-1}(1 - \lambda))] \leq \text{int} \text{cl}[(f^{-1}(1 - \lambda))] \). Then, \( \text{int} \text{cl}[(f^{-1}(1 - \lambda))] \neq 0 \). Now \( f^{-1}(1 - \lambda) \leq (1 - \lambda) \) implies that \( \text{int} \text{cl}(1 - \lambda) \neq 0 \) (A). Since \( 1 - \lambda \) is a fuzzy closed set, from (A), we have \( \text{int}(1 - \lambda) \neq 0 \), which implies that \( 1 - \text{cl}(\lambda) \neq 0 \). That is, \( \text{cl}(\lambda) \neq 1 \), a contradiction to \( \lambda \) being a fuzzy dense set in \((Y,S)\). Hence we must have \( \text{cl}[f^{-1}(\lambda)] = 1 \) and therefore that \( f^{-1}(\lambda) \) is a fuzzy dense set in \((X,T)\).
(2) $\Rightarrow$ (3) Assume (2). Suppose that $\lambda$ is a fuzzy nowhere dense in $(Y, S)$. Then $\text{int cl} \{\lambda\} = 0$. Now $1 - \text{int cl}(\lambda) = 1 - 0 = 1$, implies that $\text{cl int}[1 - \lambda] = 1$. Hence $\text{int}[1 - \lambda]$ is a fuzzy open and fuzzy dense set in $(Y, S)$. By hypothesis, $f^{-1}(\text{int}[1 - \lambda])$ is a fuzzy dense set in $(Y, S)$. That is, $\text{cl} \{f^{-1}(\text{int}[1 - \lambda])\} = 1$. Now $\text{int}[1 - \lambda] \subseteq [1 - \lambda]$, implies that $\text{cl} f^{-1}(\text{int}[1 - \lambda]) \subseteq \text{cl} f^{-1}([1 - \lambda])$. Then we have $1 \leq \text{cl} f^{-1}([1 - \lambda])$. That is, $\text{cl} f^{-1}([1 - \lambda]) = 1$, which implies that $1 - \text{int} f^{-1}(\lambda) = 1$. Therefore $\text{int} f^{-1}(\lambda) = 0$ in $(X, T)$.

(3) $\Rightarrow$ (1). Assume (3). Suppose that $\lambda$ is a non-zero fuzzy open in $(X, T)$. We claim that $\text{int cl} f(\lambda) \neq 0$. Assume the contrary. Suppose that, $\text{int cl} f(\lambda) = 0$. This implies that $f(\lambda)$ is a fuzzy nowhere dense in $(Y, S)$. Then, by hypothesis, $\text{int cl} f^{-1}(f(\lambda)) = 0$.

But $f^{-1} f(\lambda) \geq \lambda$ implies that $\text{int cl} f^{-1}(f(\lambda)) \geq \text{int cl} \lambda$ in $(X, T)$. Then $0 \geq \text{int cl} \lambda$. That is, $\text{int cl} \lambda = 0$, which implies that $\lambda = 0$ (since $\lambda$ is fuzzy open in $(X, T)$, $\text{int cl} \lambda = \lambda$). But this is a contradiction to $\lambda$ being a non-zero fuzzy (open) set in $(X, T)$. Hence we must have $\text{int cl} f(\lambda) \neq 0$. Therefore $f$ is a somewhat fuzzy nearly open function from a fuzzy topological space $(X, T)$ into the fuzzy topological space $(Y, S)$.

**Proposition 4.5.** Let $f : (X, T) \rightarrow (Y, S)$ be a fuzzy irresolute function from a fuzzy topological space $(X, T)$ into a fuzzy topological space $(Y, S)$. Then, the following are equivalent:

1. $f$ is somewhat fuzzy nearly open.

2. If $\lambda$ is a fuzzy nowhere dense set in $(Y, S)$, then $f^{-1}(\lambda)$ is a fuzzy nowhere dense set in $(X, T)$.

**Proof.** (1) $\Rightarrow$ (2) Assume (1). Suppose that $\lambda$ is a fuzzy nowhere dense set in $(Y, S)$. Then by Theorem 3.1, $\lambda$ is a fuzzy semi-closed set in $(Y, S)$. Hence $1 - \lambda$ is a fuzzy semi-open set in $(Y, S)$. Since the function $f$ is a fuzzy irresolute function, $f^{-1}(1 - \lambda)$ is a fuzzy semi-open set in $(X, T)$. Then we have $f^{-1}(1 - \lambda) \leq \text{cl int}[f^{-1}(1 - \lambda)]$. Then we have $1 - f^{-1}(\lambda) \leq \text{cl int}[1 - f^{-1}(\lambda)]$, which implies that $1 - f^{-1}(\lambda) \leq 1 - \text{int cl}[f^{-1}(\lambda)]$. Then, $\text{int cl}[f^{-1}(\lambda) \leq f^{-1}(\lambda)$. Now $\text{int cl}[f^{-1}(\lambda)] \leq \text{int cl}[f^{-1}(\lambda)]$ implies that $\text{int cl}[f^{-1}(\lambda)] = \text{int cl}[f^{-1}(\lambda)]$ (since $\text{int cl}[f^{-1}(\lambda)]$ is fuzzy open in $(X, T)$, $\text{int cl}[f^{-1}(\lambda)]$). Then, we have $\text{int cl}[f^{-1}(\lambda)] \leq \text{int cl}[f^{-1}(\lambda)]$ (B). Since $f$ is a somewhat fuzzy nearly open function and $\lambda$ is a fuzzy nowhere dense set in $(Y, S)$, by Proposition 4.4, $\text{int cl}[f^{-1}(\lambda)] = 0$ in $(X, T)$. Hence from (B), we have,
\( \text{int cl} \left[ f^{-1}(\lambda) \right] \leq 0. \) That is, \( \text{int cl} \left[ f^{-1}(\lambda) \right] = 0 \) in \( (X, T) \). Therefore \( f^{-1}(\lambda) \) is a fuzzy nowhere dense set in \( (X, T) \).

(2) \( \Rightarrow \) (1). Assume (2). Suppose that \( \lambda \) is a fuzzy open and fuzzy dense set in \( (Y, S) \). Then \( \text{int}(\lambda) = \lambda \) and \( \text{cl}(\lambda) = 1 \), implies that \( \text{cl}(\lambda) = 1 \). Then we have \( 1 - \text{cl}(\lambda) = 0 \). This implies that \( \text{int cl}(1 - \lambda) = 0 \). Hence \( 1 - \lambda \) is a fuzzy nowhere dense set in \( (Y, S) \).

By hypothesis, \( f^{-1}(1 - \lambda) \) is a fuzzy nowhere dense set in \( (X, T) \). Then, \( \text{int cl} \left[ f^{-1}(1 - \lambda) \right] = 0 \), implies that \( 1 - \text{cl}(f^{-1}(\lambda)) = 0 \). Since \( \text{int}(f^{-1}(\lambda)) \leq \text{cl}(f^{-1}(\lambda)) \), then \( 1 - \text{cl}(f^{-1}(\lambda)) \leq 1 - \text{cl}( \text{int}(f^{-1}(\lambda)) ) \). Then, we have \( 1 - \text{cl}(f^{-1}(\lambda)) \leq 0 \). That is, \( 1 - \text{cl}(f^{-1}(\lambda)) = 0 \) implies that \( \text{cl}(f^{-1}(\lambda)) = 1 \). Hence \( f^{-1}(\lambda) \) is a fuzzy dense set in \( (X, T) \). Thus, for the fuzzy open and fuzzy dense set \( \lambda \) in \( (Y, S) \), we have \( \text{cl}(f^{-1}(\lambda)) = 1 \) in \( (X, T) \). Therefore, by Proposition 4.3, \( f \) is a somewhat fuzzy nearly open function from \( (X, T) \) into \( (Y, S) \).

**Proposition 4.6.** Let \( f : (X, T) \rightarrow (Y, S) \) be a fuzzy irresolute and somewhat fuzzy nearly open function from a fuzzy topological space \( (X, T) \) into the fuzzy topological space \( (Y, S) \). If \( \lambda \) is a fuzzy first category in \( (Y, S) \), then \( f^{-1}(\lambda) \) is a fuzzy first category in \( (X, T) \).

**Proof.** Let \( \lambda \) be a fuzzy first category in \( (Y, S) \). Then \( \lambda = \bigvee_{\lambda} \left( \lambda \right) \), where \( \lambda \)'s are fuzzy nowhere dense sets in \( (Y, S) \). Now \( f^{-1}(\lambda) = f^{-1}(\bigvee_{\lambda} \left( \lambda \right)) = \bigvee_{\lambda} f^{-1}(\lambda) \). Since \( f \) is a fuzzy irresolute function and somewhat fuzzy nearly open from \( (X, T) \) into \( (Y, S) \) and \( \lambda \)'s are fuzzy nowhere dense sets in \( (Y, S) \), by Proposition 4.4, \( f^{-1}(\lambda) \)'s are fuzzy nowhere dense sets in \( (X, T) \). Hence we have \( f^{-1}(\lambda) = \bigvee_{\lambda} f^{-1}(\lambda) \), where \( f^{-1}(\lambda) \)'s are fuzzy nowhere dense sets in \( (Y, S) \). Hence \( f^{-1}(\lambda) \) is a fuzzy first category in \( (X, T) \).

5. Baire spaces and functions

**Definition 5.1** [10]. Let \( (X, T) \) be a fuzzy topological space. Then \( (X, T) \) is called a fuzzy Baire space if \( \text{int} \left( \bigvee_{\lambda} \left( \lambda \right) \right) = 0 \), where \( \lambda \)'s are fuzzy nowhere dense sets in \( (X, T) \).

**Theorem 5.1** [10]. Let \( (X, T) \) be a fuzzy topological space. Then the following are equivalent:
(1) \((X,T)\) is a fuzzy baire space.

(2) \(\text{Int}(\lambda) = 0\) for every fuzzy first category set \(\lambda\) in \((X,T)\).

(3) \(\text{cl}(\mu) = 1\) for every fuzzy residual set \(\mu\) in \((X,T)\).

**Proposition 5.1.** Let \(f : (X,T) \to (Y,S)\) be a fuzzy irresolute and somewhat fuzzy nearly open function from a fuzzy topological space \((X,T)\) into a fuzzy topological space \((Y,S)\). If \((X,T)\) is a fuzzy Baire space, then \((Y,S)\) is a fuzzy Baire space.

**Proof.** Let \(\lambda\) be a fuzzy first category in \((Y,S)\). Now \(\text{Int}(\lambda)\) is a fuzzy open set in \((Y,S)\). Since every fuzzy open set is fuzzy semi-open in a fuzzy topological space, \(\text{Int}(\lambda)\) is a fuzzy semi-open set in \((Y,S)\). Since \(f\) is a fuzzy irresolute function from \((X,T)\) into \((Y,S)\), \(f^{-1}(\text{Int}(\lambda))\) is a fuzzy semi-open set in \((X,T)\). Then, \(\text{Int}(f^{-1}(\lambda)) \subseteq \text{Cln}(f^{-1}(\lambda))\) (A). Since \(\lambda\) is a fuzzy first category in \((Y,S)\), by Proposition 4.4, \(f^{-1}(\lambda)\) is a fuzzy first category in \((X,T)\). Also since \((X,T)\) is a fuzzy Baire space, by Theorem 5.1, \(\text{Int}(f^{-1}(\lambda)) = 0\). Then, \(\text{Cln}(f^{-1}(\lambda)) = 0\). Hence from (A), we have \(f^{-1}(\lambda) = 0\). This implies that \(\text{Int}(\lambda) = 0\). Therefore by Theorem 5.1, \((Y,S)\) is a fuzzy Baire space.

**Proposition 5.2.** Let \(f : (X,T) \to (Y,S)\) be a one-to-one, fuzzy pre semi-open and somewhat fuzzy nearly continuous function from a fuzzy topological space \((X,T)\) onto the fuzzy topological space \((Y,S)\). If \((Y,S)\) is a fuzzy Baire space, then \((X,T)\) is a fuzzy Baire space.

**Proof.** Let \(\lambda\) be a fuzzy first category in \((X,T)\). Now \(\text{Int}(\lambda)\) is a fuzzy open set in \((X,T)\). Since every fuzzy open set is fuzzy semi-open in a fuzzy topological space, \(\text{Int}(\lambda)\) is a fuzzy semi-open set in \((X,T)\). Since \(f\) is a fuzzy pre semi-open function from \((X,T)\) onto \((Y,S)\), \(f[\text{Int}(\lambda)]\) is a fuzzy semi-open set in \((Y,S)\). Then, \(f[\text{Int}(\lambda)] \subseteq \text{Cln}(f[\text{Int}(\lambda)])\). Then, \(f[\text{Int}(\lambda)] \subseteq \text{Cln}(f[\text{Int}(\lambda)])\) (A). Since \(\lambda\) is a fuzzy first category in \((X,T)\), by Proposition 3.4, \(f(\lambda)\) is a fuzzy first category in \((Y,S)\). Also since \((Y,S)\) is a fuzzy Baire space, by Theorem 5.1, \(f[f(\lambda)] = 0\). Then, \(\text{Cln}[f(\lambda)] = \text{Cl}(0) = 0\). Hence from (A), we have \(f[f(\lambda)] = 0\). This implies that \(\text{Int}(\lambda) = 0\). Therefore by Theorem 5.1, \((Y,S)\) is a fuzzy Baire space.

**References**


Abstract:

In this paper, some new common fixed point theorems under certain strict contractive conditions for mappings sharing the $(CLR_{ST})$ property in fuzzy metric spaces are proved. Examples in support of our results are also given. A fixed point theorem for four finite families of self mappings in fuzzy metric space is also obtained. Our results improve and extend the results of Sedghi and Shobe [Common fixed point theorems under strict contractive conditions in fuzzy metric spaces using property (E.A). Commun. Korean Math. Soc. 27 (2), 399-410 (2012)].

Key words and phrases:

Fuzzy metric space; weakly compatible mappings; property (E. A); common property (E. A); common limit range property; fixed point.

1. Introduction
Probability theory provides some tools to measure uncertainty in our real world of activities. In fact, the more general (i.e., not necessarily probabilistic in nature) concept of “uncertainty” is considered a basic ingredient of some basic mathematical structures. The concept of fuzzy sets [1] constitutes an example, where the concept of uncertainty was introduced in the theory of sets, in a non probabilistic manner. Fuzzy set theory has applications in applied sciences such as mathematical programming, modeling theory, engineering sciences, image processing, control theory, communication etc. In 1975, Kramosil and Michalek [2] introduced the concept of fuzzy metric space as a generalization of the statistical (probabilistic) metric space. In fact the study of such spaces received an impetus with the pioneering work of Heilpern [3]. Afterwards, Grabiec [4] defined the completeness of the fuzzy metric space and extended the Banach contraction principle to fuzzy metric spaces. Further, Fang [5] established some new fixed point theorems for contractive type mappings in fuzzy metric spaces. Soon after, Mishra et al. [6] also proved several fixed point theorems for asymptotically commuting mappings in such spaces. Since then, the notion of a complete fuzzy metric space presented by George and Veeramani [7] which constitutes a modification of the one due to Kramosil and Michalek [2]. Many authors have contributed to the development of this theory and apply to fixed point theory, for instance [8,9,10,11,12].

Mishra et al. [6] extended the notion of compatible mappings (introduced by Jungck [13] in metric space) to fuzzy metric spaces and proved common fixed point theorems in presence of continuity of at least one of the mappings, completeness of the underlying space and containment of the ranges amongst involved mappings. Further, Singh and Jain [14] (introduced by Jungck and Rhoades [15] in metric space) weakened the notion of compatibility by using the notion of weakly compatible mappings in fuzzy metric spaces and showed that every pair of compatible mappings is weakly compatible but converse is not true. However, the study of common fixed points of non-compatible mappings (introduced by Pant [16] in metric space) in fuzzy metric space is also of great interest due to Pant and Pant [17]. In 2009, Abbas et al. [18] proved a common fixed point theorem for two pairs of weakly compatible mappings in fuzzy metric space by using the property (E. A) (introduced by Aamri and El-Moutawakil [19] in metric space). They also utilized the notion of common property property (E. A) (introduced by Liu et al. [20] in metric space) to prove the same result. It is observed that the notion of common property (E. A) relatively relaxes the required containment of the range of one mapping into the range of another which is utilized to construct the sequence of joint iterates (see [21]). In recent past, several authors proved various fixed point theorems employing relatively more general contractive conditions (e.g., [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43]).

Recently, Sintunavarat and Kumam [44] coined the idea of “common limit range property” in fuzzy metric spaces. They showed that common limit range property never requires the closedness of the subspace (also see [45]) for the existence of fixed point. Most recently, Sedghi and Shobe [46] proved a common fixed point theorem for weakly compatible mappings satisfying strict contractive conditions in fuzzy metric spaces by using the property (E. A).

The aim of this paper is to prove a common fixed point theorem for two pairs of weakly compatible mappings in fuzzy metric space by using common limit range property. Some examples are proved to validate our results. We extend our main result to four finite families of mappings by using the notion of pairwise commuting property of two finite families of mappings due to Imdad et al. [47].
2. Preliminaries

Definition 2.1 [1]. Let $X$ be any set. A fuzzy set $A$ in $X$ is a function with domain $X$ and values in $[0, 1]$.

Let $\mathcal{F}$ be the collection of all fuzzy sets on $X^2 \times [0, \infty)$, that is, $\mathcal{F} = \{f : X^2 \times [0, \infty) \rightarrow [0, 1]\}$.

Definition 2.2 [46]. Let $f, g \in \mathcal{F}$. The algebraic sum $f \oplus g$ of $f$ and $g$ is defined by

$$f(x, y, t) \oplus g(x', y', t) = \sup_{t_1 + t_2 = t} \min \{f(x, y, t_1), g(x, y, t_2)\}$$

Remark 2.1 [46]. For every $x, y \in X$ and every $t > 0$, we have

1. $f(x, y, 2t) \oplus f(x, y, 2t) \geq \min \{f(x, y, t) + t, f(x, y, t)\}$
2. $f(x, y, t + \varepsilon) \oplus f(x, x, \varepsilon) = f(x, y, t + \varepsilon) + f(x, x, \varepsilon)$

Letting $\varepsilon \rightarrow 0$, we get $f(x, y, t) \oplus 1 \geq f(x, y, t)$. Throughout this paper $\Phi$ denotes a family of mappings such that for each $\phi \in \Phi$, $\phi : [0, 1] \rightarrow [0, 1]$ is continuous and increasing in each co-ordinate variable. Also $\gamma(t) = \phi(t, t, t) > t$ for every $t \in [0, 1]$.

Example 2.1 [46]. Let $\phi : [0, 1]^3 \rightarrow [0, 1]$ be defined by

$$\phi(x, y, z) = \min \{x, y, z\}.$$

Definition 2.3 [48]. A mapping $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous $t$-norm if $([0, 1], *)$ is an abelian topological monoid with unit 1 such that $a \ast b \leq c \ast d$, for $a \leq c$, $b \leq d$. Three typical examples of $t$-norms are $a * b = \min \{a, b\}$ (minimum $t$-norm), $a * b = ab$ (product $t$-norm), and $a * b = \max \{a + b - 1, 0\}$ (Lukasiewicz $t$-norm).

Definition 2.4 [2]. The 3-tuple $(X, M, \ast)$ is called a fuzzy metric space if $X$ is an arbitrary set, $\ast$ is a continuous $t$-norm and $M$ is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:

(a) $M(x, y, t) > 0$,
(b) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(c) $M(x, y, t) = M(y, x, t)$,
(d) $M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)$,
(e) $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is continuous,

for each $x, y, z \in X$ and $t, s > 0$. 

Definition 2.5. Then $M$ is called a fuzzy metric on $X$. Note that, $M(x, y, t)$ can be thought of as the definition of nearness between $x$ and $y$ with respect to $t$. It is known that $M(x, y, t)$ is nondecreasing for all $x, y \in X$ [4]. Throughout this work, $X$ will represent a fuzzy metric space equipped with a continuous $t$-norm and a fuzzy set $M$ until or unless stated otherwise.

From the following example, it is assured that every metric induces a fuzzy metric.

Example 2.2 [7]. Let $(X, d)$ be a metric space. Denote $a*b = ab$ (or $a*b = \min\{a, b\}$) for all $a, b \in [0, 1]$ and let $M_d$ be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows: $M_d(x, y, t) = tf(t + d(x, y))$. Then $(X, M_d, *)$ is a fuzzy metric space and the fuzzy metric $M$ induced by the metric $d$ is often referred to as the standard fuzzy metric.

Definition 2.6 [7]. A sequence $\{x_n\}$ in $X$ is said to be:

1. convergent to some $x$ in $X$ if for each $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$; i.e., $M(x_n, x, t) \to 1$ as $n \to \infty$ for all $t > 0$.

2. Cauchy if for each $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$; i.e., $M(x_n, x_m, t) \to 1$ as $n, m \to \infty$ for all $t > 0$.

A fuzzy metric space $X$ in which every Cauchy sequence is convergent is said to be complete.

Definition 2.7 [4]. A fuzzy set $M$ is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n \to \infty} M(x_n, y_n, t_n) = M(x, y, t)$$

whenever $(x_n, y_n, t_n)$ is a sequence in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$; i.e.,

$$\lim_{n \to \infty} M(x_n, x, t) = \lim_{n \to \infty} M(y_n, y, t) = 1, \text{ and } \lim_{n \to \infty} M(x_n, y, t_n) = M(x, y, t).$$

Definition 2.8 [18]. Two self mappings $A$ and $S$ on $X$ are said to be:

(i) compatible if and only if $M(ASx_n, SAx_n, t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in $X$ such that $Ax_n, Sx_n \to z$ for some $z \in X$ as $n \to \infty$ [6].

(ii) non-compatible, if there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = z$ for some $z \in X$, but for some $t > 0$, $\lim_{n \to \infty} M(ASx_n, SAx_n, t) = \text{either less than 1 or nonexistent.}$

(iii) weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Az = Bz$ for some $z \in X$, then $ABz = BAz$ [15].

It is known that a pair $(A, S)$ of compatible mappings is weakly compatible but converse is not true in general.
Definition 2.9. A pair of self mappings \((A, S)\) defined on a fuzzy metric space \(X\) is said to satisfy:

(iv) the property (E. A) if there exists a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = z\), for some \(z \in X\) [18].

(v) the common limit range property with respect to mapping \(S\) (briefly, \((CLR_s)\) property) if there exists a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = z\), where \(z \in S(X)\) [44].

It is straightforward to notice that every pair of non-compatible self mappings of a fuzzy metric space \(X\) satisfies the property (E. A) but not conversely (see [5, Example 1]). Here, it can be pointed out that weakly compatibility and the property (E. A) are independent to each other (see [49, Example 2.1]). It is evident that a pair \((A, S)\) satisfying the property (E. A) along with closedness of the subspace \(S(X)\) always enjoys the \((CLR_s)\) property.

With a view to extend the \((CLR_s)\) property to two pair of self mappings, we defined the \((CLR_{ST})\) property with respect to mappings \(S\) and \(T\) in the following definition.

Definition 2.10. Two pairs of self mappings \((A, S)\) and \((B, T)\) of a fuzzy metric space \(X\) are said to satisfy

(vi) the common property (E. A), if there exist two sequences \(\{x_n\}, \{y_n\}\) in \(X\) for some \(z\) in \(X\) such that \(\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = z\) [18].

(vii) The common limit range property with respect to mappings \(S\) and \(T\) (briefly, \((CLR_{ST})\) property) if there exist two sequences \(\{x_n\}, \{y_n\}\) in \(X\) such that

\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = z, \]

where \(z \in S(X) \cap T(X)\).

Example 2.3. Let \((X, M, \ast)\) be a fuzzy metric space, where \(X = [3, 20]\), with \(t\) - norm \(\ast\) is defined by \(a \ast b = \min\{a, b\}\) for all \(a, b \in [0, 1]\) and \(M(x, y, t) = t/(t + |x - y|)\), if \(t > 0\); for all \(x, y \in X\) and \(t > 0\). Define the self mappings \(A, B, S, T\) by

\[
A(x) = \begin{cases} 
7, & \text{if } x = 3; \\
5, & \text{if } 3 < x \leq 14; \\
(x + 1)/5, & \text{if } x > 14.
\end{cases} 
\]

\[
B(x) = \begin{cases} 
4, & \text{if } x = 3; \\
(4x + 3)/5, & \text{if } 3 < x \leq 14; \\
13, & \text{if } x > 14.
\end{cases} 
\]

\[
S(x) = \begin{cases} 
5, & \text{if } x = 3; \\
15, & \text{if } 3 < x \leq 14; \\
(2x - 1)/9, & \text{if } x > 14.
\end{cases} 
\]

\[
T(x) = \begin{cases} 
6, & \text{if } x = 3; \\
(x + 3)/2, & \text{if } 3 < x \leq 14; \\
17, & \text{if } x > 14.
\end{cases} 
\]

If we choose two sequence as \(\{x_n\} = \{14 + 1/n\}_{n \in \mathbb{N}}\) and \(\{y_n\} = \{3 + 1/n\}_{n \in \mathbb{N}}\), then the pairs \((A, S)\) and
$(B, T)$ enjoy the common property (E. A) for all $t > 0$: \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = 3 \in X \). Here it is noticed that $3 \not\in S(X) \cap T(X)$. Therefore, the pairs $(A, S)$ and $(B, T)$ do not satisfy the common limit range property with respect to mappings $S$ and $T$.

In view of Example 2.3, the following proposition is predictable.

**Proposition 2.1.** If the pairs $(A, S)$ and $(B, T)$ enjoy the common property (E. A) and $S(X)$ as well as $T(X)$ are closed subsets of $X$, then the pairs also enjoy the $(CLR_{ST})$ property.

The following definition is essentially contained in Imdad et al. [47].

**Definition 2.11.** Two families of self mappings \( \{A_i\}_{i=1}^m \) and \( \{S_k\}_{k=1}^n \) are said to be pairwise commuting if

1. \( A_iA_j = A_jA_i \) for all $i, j \in \{1, 2, \ldots, m\}$,
2. \( S_kS_l = S_lS_k \) for all $k, l \in \{1, 2, \ldots, n\}$,
3. \( A_iS_k = S_kA_i \) for all $i \in \{1, 2, \ldots, m\}$ and $k \in \{1, 2, \ldots, n\}$.

3. Results

In this section, it is assumed that $M$ is a continuous fuzzy metric on $X^2 \times (0, \infty)$ such that $\lim_{t \to \infty} M(x, y, t) = 1$, for all $x, y \in X$.

Sedghi and Shobe [46] proved the following fixed point theorem for two pairs of weakly compatible mappings satisfying strict contractive condition in fuzzy metric spaces using (E. A) property.

**Theorem 3.1** [46, Theorem 2.4]. Let $A, B, S$ and $T$ be self mappings on a fuzzy metric space $X$ satisfying the following conditions:

1. \( A(X) \subseteq T(X), \ B(X) \subseteq S(X) \).
2. \[
M(Ax, By, t) \geq \phi \left( M(Sx, Ty, 2t/k), M(Ax, Sx, 2t/k) \oplus M(By, Ty, 2t/k), M(Ax, Ty, 4t/k) \oplus M(Sx, By, 4t/k) \right),
\]

for all $x, y \in X$, $t > 0$, $\phi \in \Phi$ and $0 \leq k < 2$. Suppose that one of the pairs $(A, S)$ and $(B, T)$ satisfies the (E. A) property, $(A, S)$ and $(B, T)$ are weakly compatible and one of $A(X), B(X), S(X)$ and $T(X)$ is a complete subspace of $X$. Then $A, B, S$ and $T$ have a unique common fixed point in $X$.

**Lemma 3.1.** Let $A, B, S$ and $T$ be self mappings of a fuzzy metric space $X$ satisfying inequality (3.1) of Theorem 3.1. Suppose that
The pair \((A, S)\) satisfies the \(\text{CLR}_S\) property (or the pair \((B, T)\) satisfies the \(\text{CLR}_T\) property).

(2) \(A(X) \subset T(X)\) (or \(B(X) \subset S(X)\)).

(3) \(T(X)\) (or \(S(X)\)) is a closed subset of \(X\).

(4) \(B(y_n)\) converges for every sequence \(\{y_n\}\) in \(X\) whenever \(T(y_n)\) converges (or \(A(x_n)\) converges for every sequence \(\{x_n\}\) in \(X\) whenever \(S(x_n)\) converges).

Then the pairs \((A, S)\) and \((B, T)\) share the \(\text{CLR}_{ST}\) property.

Proof. If pair \((A, S)\) satisfies the \(\text{CLR}_S\) property, then there exists a sequence \(\{x_n\}\) in \(X\) such that
\[
\lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n = z, \tag{3.2}
\]
where \(z \in S(X)\). As \(A(X) \subset T(X)\), for a sequence \(\{x_n\} \subset X\) there corresponds a sequence \(\{y_n\} \subset X\) such that \(A x_n = T y_n\). Therefore,
\[
\lim_{n \to \infty} T y_n = \lim_{n \to \infty} A x_n = z, \tag{3.3}
\]
where \(z \in S(X) \cap T(X)\). Thus, we have \(A x_n \to z\), \(S x_n \to z\) and \(T y_n \to z\). Now we claim that \(B y_n \to z\). Using (3.1) with \(x = x_n\), \(y = y_n\), we get
\[
M(A x_n, B y_n, t) \geq \phi \left( \frac{M(S x_n, T y_n, 2t/k)}{M(A x_n, T y_n, 4t/k)} \right), \tag{3.4}
\]
Since 
\[
\liminf_{n \to \infty} \left\{ M(A x_n, S x_n, 2t/k) \right\} \geq \liminf_{n \to \infty} \left\{ M(A x_n, S x_n, \epsilon) \right\},
\]
and
\[
\liminf_{n \to \infty} \left\{ M(A x_n, S x_n, 2t/k - \epsilon) \right\} = \liminf_{n \to \infty} M(B y_n, T y_n, 2t/k - \epsilon). \]

Letting \(\epsilon \to 0\), in above inequality, we obtain
\[
\liminf_{n \to \infty} \left\{ M(A x_n, S x_n, 2t/k) \right\} \geq \liminf_{n \to \infty} M(B y_n, T y_n, 2t/k). \]

Also, \(\liminf_{n \to \infty} \left\{ M(A x_n, T y_n, 4t/k) \right\} \geq \liminf_{n \to \infty} M(B y_n, S x_n, 4t/k)\) passing to limit as \(n \to \infty\) in inequality (3.4), we obtain
\[
\liminf_{n \to \infty} M(A x_n, B y_n, t) \geq \phi \left( \liminf_{n \to \infty} \left\{ M(S x_n, T y_n, 2t/k) \right\} \right),
\]
where
\[
\liminf_{n \to \infty} \left\{ M(A x_n, S x_n, 2t/k) \right\}, \liminf_{n \to \infty} \left\{ M(A x_n, T y_n, 4t/k) \right\}, \liminf_{n \to \infty} \left\{ M(S x_n, B y_n, 2t/k) \right\}, \liminf_{n \to \infty} \left\{ M(S x_n, B y_n, 4t/k) \right\}.
\]
\[
\geq \phi \left( \lim_{n \to \infty} \inf M \left( Sx_n, Ty_n, 2t/k \right), \lim_{n \to \infty} \inf M \left( By_n, Ty_n, 2t/k \right), \right),
\]
and so
\[
M \left( z, \lim_{n \to \infty} \inf B y_n, t \right) \geq \phi \left( M \left( z, z, 2t/k \right), M \left( \lim_{n \to \infty} \inf B y_n, z, 2t/k \right), \right)
\]
\[
\geq \phi \left( M \left( \lim_{n \to \infty} \inf B y_n, z, 2t/k \right), M \left( \lim_{n \to \infty} \inf B y_n, z, 2t/k \right), \right) \geq M \left( z, \lim_{n \to \infty} \inf B y_n, 2t/k \right)
\]
\[
\geq M \left( z, \lim_{n \to \infty} \inf B y_n, \left( 2t/k \right)^n \right) \to 1 \quad \ldots.
\]
Similarly, \( \limsup_{n \to \infty} M \left( z, By_n, t \right) = M \left( z, \limsup_{n \to \infty} B y_n, t \right). \) Hence, \( \lim_{n \to \infty} M \left( z, By_n, t \right) = 1. \)
It implies \( By_n \to z \) as \( n \to \infty. \) Therefore the pairs \((A, S)\) and \((B, T)\) share the \((CLR_{ST})\) property.

**Theorem 3.2.** Let \( A, B, S, \) and \( T \) be soft mappings of a fuzzy metric space \((X, M, *)\)
satisfying inequality (3.1) of Theorem 3.1 If pair \((A, S)\) and \((B, T)\) enjoy the \((CLR_{ST})\) property, then \((A, S)\) and \((B, T)\) have a coincidence point each. Moreover, \( A, B, S \) and \( T \) have a unique common fixed point provided that both the pairs \((A, S)\) and \((B, T)\) are weakly compatible.

**Proof.** Since the pairs \((A, S)\) and \((B, T)\) share the \((CLR_{ST})\) property, there exist two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ty_n = \lim_{n \to \infty} By_n = z, \) where \( z \in S(X) \cap T(X). \) Since \( z \in S(X), \) there exists a point \( u \in X \) such that \( Su = z. \) Now we assert that \( Au = Su. \) Using inequality (3.1) with \( x = u, y = y_n, \) we obtain
\[
M \left( Au, By_n, t \right) \geq \phi \left( M \left( Su, Ty_n, 2t/k \right), M \left( Au, Su, 2t/k \right) \oplus M \left( By_n, Ty_n, 2t/k \right), \right.
\]
\[
M \left( Au, Ty_n, 4t/k \right) \oplus M \left( Su, By_n, 4t/k \right)
\]
that is,
\[
M \left( Au, By_n, t \right) \geq \phi \left( M \left( z, Ty_n, 2t/k \right), M \left( Au, z, 2t/k \right) \oplus M \left( By_n, Ty_n, 2t/k \right), \right.
\]
\[
M \left( Au, Ty_n, 4t/k \right) \oplus M \left( z, By_n, 4t/k \right)
\]
(3.5)
Since \( \liminf_{n \to \infty} \left[ M \left( Au, z, 2t/k \right) \oplus M \left( By_n, Ty_n, 2t/k \right) \right] \geq \liminf_{n \to \infty} \min \left\{ M \left( Au, z, 2t/k - \epsilon \right), \right. \)
\[
M \left( By_n, Ty_n, \epsilon \right) \right\} \text{ for all } \epsilon \in (0, 2t/k). \) Letting \( \epsilon \to 0 \) in above inequality, we get
\[
\liminf_{n \to \infty} \left[ M \left( Au, z, 2t/k \right) \oplus M \left( By_n, Ty_n, 2t/k \right) \right] \geq \liminf_{n \to \infty} M \left( Au, z, 2t/k \right). \) Also \( \liminf_{n \to \infty} \left\{ M \left( Au, Ty_n, 4t/k \right) \oplus M \left( z, By_n, 4t/k \right) \right\} = \liminf_{n \to \infty} M \left( Au, Ty_n, 4t/k \right) \oplus 1 \geq M \left( Au, z, 2t/k \right) \)


$4t/k \geq M(A_u, z, 2t/k)$. Now passing to limit as $n \to \infty$ in inequality (3.5), we obtain

\[
M(A_u, z, t) \geq \phi(1, M(A_u, z, 2t/k), M(A_u, z, 2t/k)) \geq \phi(M(A_u, z, 2t/k), M(A_u, z, 2t/k), M(A_u, z, 2t/k)) \geq M(A_u, z, 2t/k) \quad \cdots \geq M\left(A_u, z, (2/k)^n t \right) \to 1.
\]

Hence, $M(A_u, z, t) = 1$, that is, $A_u = Su = z$ and hence $u$ is a coincidence point of $(A, S)$. Also $z \in T(X)$, there exists a point $v \in X$ such that $Tv = z$. Now we show that $Bv = Tv$. On using inequality (3.1) with $x = u$, $y = v$, we get

\[
M(A_u, B_u, v) \geq \phi\left(M(S_u, Tv, 2t/k), M(A_u, Su, 2t/k) \oplus M(Bv, Tv, 2t/k), \frac{M(A_u, Tv, 4t/k) \oplus M(S_u, Bv, 4t/k)}{M(A_u, B_u, v) \oplus M(S_u, Bv, 4t/k)}\right)
\]

\[
M(z, B_v, t) \geq \phi(1, 1 \oplus M(Bv, z, 2t/k), 1 \oplus M(z, Bv, 4t/k)) \geq \phi(1, M(Bv, z, 2t/k), M(z, Bv, 4t/k)) \geq \phi(M(Bv, z, 2t/k), M(Bv, z, 2t/k), M(z, Bv, 2t/k)) \geq M(z, Bv, 2t/k) \quad \cdots \geq M\left(Bv, z, (2/k)^n t \right) \to 1.
\]

Hence $z = Bv$. Therefore $Bv = Tv = z$ and hence $v$ is a coincidence point of $(B, T)$. Since the pair $(A, S)$ is weakly compatible, therefore $Az = ASz = ASAu = Sz$. Taking $x = z$ and $y = w$ in inequality (3.1), we have

\[
M(A_z, B_v, t) \geq \phi\left(M(S_z, Tv, 2t/k), M(A_z, Sz, 2t/k) \oplus M(Bv, Tv, 2t/k), \frac{M(A_z, Tv, 4t/k) \oplus M(S_z, Bv, 4t/k)}{M(A_z, B_v, t) \oplus M(S_z, Bv, 4t/k)}\right)
\]

\[
M(A_z, z, t) \geq \phi(M(A_z, z, 2t/k), 1, M(A_z, z, 4t/k)) \geq \phi(M(A_z, z, 2t/k), M(A_z, z, 4t/k), M(A_z, z, 2t/k)) \geq M(A_z, z, 2t/k) \cdots \geq M\left(A_z, z, (2/k)^n t \right) \to 1,
\]

which implies that $Az = z = Sz$, therefore $z$ is a common fixed point of the pair $(A, S)$. Also the pair $(B, T)$ is weakly compatible, therefore $Bz = BTv = TBv = Tz$. On using inequality (3.1) with $x = u$, $y = z$, we have

\[
M(A_u, B_z, t) \geq \phi\left(M(S_u, Tz, 2t/k), M(A_u, Su, 2t/k) \oplus M(Bz, Tz, 2t/k), \frac{M(A_u, Tz, 4t/k) \oplus M(S_u, Bz, 4t/k)}{M(A_u, B_z, t) \oplus M(S_u, Bz, 4t/k)}\right)
\]

\[
M(z, B_z, t) \geq \phi(M(z, Bz, 2t/k), 1, M(z, Bz, 4t/k)) \geq \phi(M(z, Bz, 2t/k), M(z, Bz, 2t/k), M(z, Bz, 2t/k)) \geq M(z, \cdots \geq M\left(z, Bz, (2/k)^n t \right) \to 1,
\]

which implies that $Bz = z$. Therefore, $Bz = z = Tz$. Hence $z$ is a common fixed point of both the pairs $(A, S)$ and $(B, T)$. The uniqueness of common fixed point is an easy consequence of inequality (3.1).

Following example illustrates Theorem 3.2.
Example 3.1. Let \((X, M, \ast)\) be a fuzzy metric space, where \(X = [5, 25]\), with \(t\)-norm \(\ast\) is defined by \(a \ast b = \min\{a, b\}\) for all \(a, b \in [0, 1]\) and
\[
M(x, y, t) = \begin{cases} 
\frac{t}{t + |x - y|}, & \text{if } t > 0; \\
0, & \text{if } t = 0
\end{cases}
\]
for all \(x, y \in X\) and \(t > 0\). Let \(\phi : [0, 1] \rightarrow [0, 1]\) be the same as given in Example 2.1. Define self mappings \(A, B, S\) and \(T\) by
\[
A(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
22, & \text{if } x \in (5, 10].
\end{cases}
\]
\[
B(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
12, & \text{if } x \in (5, 10].
\end{cases}
\]
\[
S(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
11, & \text{if } x \in (5, 10]; \\
(x+5)/3, & \text{if } x \in (10, 25).
\end{cases}
\]
\[
T(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
13, & \text{if } x \in (5, 10]; \\
x - 5, & \text{if } x \in (10, 25).
\end{cases}
\]
\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = z\)
for all \(x, y \in X\) and \(t > 0\). Let \(\phi : [0, 1] \rightarrow [0, 1]\) be the same as given in Example 2.1. Define self mappings \(A, B, S\) and \(T\) by
\[
A(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
22, & \text{if } x \in (5, 10].
\end{cases}
\]
\[
B(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
12, & \text{if } x \in (5, 10].
\end{cases}
\]
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S(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
11, & \text{if } x \in (5, 10]; \\
(x+5)/3, & \text{if } x \in (10, 25).
\end{cases}
\]
\[
T(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
13, & \text{if } x \in (5, 10]; \\
x - 5, & \text{if } x \in (10, 25).
\end{cases}
\]
for all \(x, y \in X\) and \(t > 0\). Let \(\phi : [0, 1] \rightarrow [0, 1]\) be the same as given in Example 2.1. Define self mappings \(A, B, S\) and \(T\) by
\[
A(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
22, & \text{if } x \in (5, 10].
\end{cases}
\]
\[
B(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
12, & \text{if } x \in (5, 10].
\end{cases}
\]
\[
S(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
11, & \text{if } x \in (5, 10]; \\
(x+5)/3, & \text{if } x \in (10, 25).
\end{cases}
\]
\[
T(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
13, & \text{if } x \in (5, 10]; \\
x - 5, & \text{if } x \in (10, 25).
\end{cases}
\]
for all \(x, y \in X\) and \(t > 0\). Let \(\phi : [0, 1] \rightarrow [0, 1]\) be the same as given in Example 2.1. Define self mappings \(A, B, S\) and \(T\) by
\[
A(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
22, & \text{if } x \in (5, 10].
\end{cases}
\]
\[
B(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
12, & \text{if } x \in (5, 10].
\end{cases}
\]
\[
S(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
11, & \text{if } x \in (5, 10]; \\
(x+5)/3, & \text{if } x \in (10, 25).
\end{cases}
\]
\[
T(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
13, & \text{if } x \in (5, 10]; \\
x - 5, & \text{if } x \in (10, 25).
\end{cases}
\]
for all \(x, y \in X\) and \(t > 0\). Let \(\phi : [0, 1] \rightarrow [0, 1]\) be the same as given in Example 2.1. Define self mappings \(A, B, S\) and \(T\) by
\[
A(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
22, & \text{if } x \in (5, 10].
\end{cases}
\]
\[
B(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
12, & \text{if } x \in (5, 10].
\end{cases}
\]
\[
S(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
11, & \text{if } x \in (5, 10]; \\
(x+5)/3, & \text{if } x \in (10, 25).
\end{cases}
\]
\[
T(x) = \begin{cases} 
5, & \text{if } x \in \{5\} \cup (10, 25]; \\
13, & \text{if } x \in (5, 10]; \\
x - 5, & \text{if } x \in (10, 25).
\end{cases}
\]
Clearly, both the pairs \((A, S)\) and \((B, T)\) satisfy the \((CLR_{e_f})\) property \(\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = x \in S(X) \cap T(X)\). Notice that \(A(X) = \{5, 22\} \subset [5, 23] = T(X)\) and \(B(X) = \{5, 12\} \subset [5, 13] = S(X)\). Also the remaining conditions of Theorem 3.3 can be easily verified for some fixed \(k \in [0, 2]\). Moreover 5 is the unique common fixed point of the pairs \((A, S)\) and \((B, T)\). Note that Theorem 3.2 can not be used in the context of this example as \(S(X)\) and \(T(X)\) are closed subsets of \(X\). Again mappings involved herein are discontinuous seven at their unique common fixed point.

**Remark 3.1.** The conclusions of Lemma 3.1, Theorem 3.2 and Theorem 3.3 remain true if we replace inequality (3.1) by the following:

\[
M(Ax, By, t) \geq \min \left\{ \frac{M(Sx, Ty, 2t/k)}{M(Ax, Ty, 2t/k)} \frac{M(Ax, Sx, 2t/k)}{M(By, Ty, 2t/k)} \right\},
\]

for all \(x, y \in X\), \(t > 0\) and \(0 \leq k < 2\).

By choosing \(A, B, S\) and \(T\) suitably, we can deduce corollaries involving two or three self mappings. As a sample, we obtain the following natural result for a pair of self mappings.

**Corollary 3.1.** Let \(A\) and \(S\) be self mappings of a fuzzy metric space \((X, M, *)\). Suppose that

(1) The pair \((A, S)\) satisfies the \((CLR_{e_f})\) property,

(2) \(S(X)\) is a closed subset of \(X\),

(3) \(M(Ax, Ay, t) \geq \phi \left( \frac{M(Sx, Sy, 2t/k)}{M(Ax, Ty, 2t/k)} \frac{M(Ax, Sx, 2t/k)}{M(By, Ty, 2t/k)} \right)\)

for all \(x, y \in X\), \(t > 0\), \(\phi \in \Phi\) and \(0 \leq k < 2\). Then \((A, S)\) has a coincidence point. Moreover if the pair \((A, S)\) is weakly compatible then it has a unique common fixed point in \(X\).

Now, we utilize Definition 2.11 (which is indeed a natural extension of commutativity condition to two finite families) to prove a common fixed point theorem for four finite families of weakly compatible mappings in fuzzy metric space (as an application of Theorem 3.2).

**Theorem 3.4.** Let \(\{A_i\}_{i=1}^m\), \(\{B_j\}_{j=1}^n\), \(\{S_k\}_{k=1}^p\) and \(\{T_l\}_{l=1}^q\) be four finite families of self mappings of a fuzzy metric space \((X, M, *)\) with \(A = A_1A_2\cdots A_m\), \(B = B_1B_2\cdots B_n\), \(S = S_1S_2\cdots S_p\) and \(T = T_1T_2\cdots T_q\) satisfying the inequality (3.1) of Theorem 3.1 such
that the pairs \((A,S)\) and \((B,T)\) share the \((\text{CLR}_{ST})\) property, then each \((A,S)\) and \((B,T)\) have a point of coincidence.

Then \(\{A_i\}_{i=1}^m, \{B_j\}_{j=1}^p, \{S_k\}_{k=1}^r, \text{ and } \{T_h\}_{h=1}^q\) have a unique common fixed point provided the pairs of families \(\{A_i\}, \{S_k\}\) and \(\{B_j\}, \{T_h\}\) commute pairwise wherein \(i \in \{1,2,\ldots,m\}, \ k \in \{1,2,\ldots,r\}, \ r \in \{1,2,\ldots,n\}\) and \(h \in \{1,2,\ldots,q\}\).

**Proof.** Since the proof of this theorem is similar to the result contained in [21], hence it is omitted.

By setting \(A_1 = A_2 = \cdots = A_m = A, \ B_1 = B_2 = \cdots = B_p = B, \ S_1 = S_2 = \cdots = S_r = S\) and \(T_1 = T_2 = \cdots = T_q = T\) in Theorem 3.4, we deduce the following:

**Corollary 3.2.** Let \(A, B, S\) and \(T\) be self mappings of a fuzzy metric space \((X, M, \ast)\). Suppose that

1. The pairs \(A^m, S^p\) and \(B^p, T^q\) share the \((\text{CLR}_{S^p,T^q})\) property.

2. \(M(A^m x, B^p y, t) \geq \phi\left(M(S^p x, T^q y, 2t/k), M(A^m x, S^p x, 2t/k), M(B^p y, T^q y, 2t/k), M(A^m x, T^q y, 4t/k), M(S^p x, B^p y, 4t/k)\right),\)

for all \(x, y \in X, \ t > 0, \ \phi \in \Phi, \ 0 \leq k < 2, \ m, \ n, \ p \text{ and } q\) are fixed positive integers.

Then \(A, B, S\) and \(T\) have a unique common fixed point provided \(AS = SA\) and \(BT = TB\).

**Remark 3.2.** The results similar to Theorem 3.4 and Corollary 3.2 can also be outlined in respect of inequality (3.6).

4. **Conclusion**

Theorem 3.2 is proved for two pairs of single-valued mappings in fuzzy metric spaces employing the \((\text{CLR}_{ST})\) property. It improves the results of Sedghi and Shobe [46, Theorem 2.4] in the sense that the conditions on completeness (or closedness) of the underlying space (or subspaces), containment of ranges amongst involved mappings together with conditions on continuity in respect of any one of the involved mappings are relaxed. A natural result is obtained in the form of a corollary. Theorem 3.4 is presented as an extension of Theorem 3.2 to four finite families of mappings.

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Some Fixed Point Results in Fuzzy Metric Spaces under Common Limit Range Property


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Some Fixed Point Theorems on Generalized $\mathcal{M}$-fuzzy Metric Space

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Abstract:
We define compatibility and weak compatibility of a collection of self maps on generalized $\mathcal{M}$-fuzzy metric space. This concept motivates us to establish a unique fixed point for a collection of self maps. Also we study a fixed point theorem for multivalued maps on generalized $\mathcal{M}$-fuzzy metric space.

Keywords:
Generalized $\mathcal{M}$-fuzzy metric space, self map, compatible maps and weakly compatible maps, multivalued map.

0. Introduction

After the introduction of fuzzy sets by Zadeh [25], many researchers concentrated on fuzzy Mathematics. Probabilistic metric [16] was introduced by Menger in 1947. Subsequently Kramosil and Michalek [15] introduced the concept of fuzzy metric spaces. Recently Sedghi and Shobe [20] introduced $\mathcal{M}$-fuzzy metric space. We introduced generalized $\mathcal{M}$-fuzzy metric space [21]. The main objective of this paper is to study some fixed point theorems on a generalized $\mathcal{M}$-fuzzy metric space. Also we define compatibility and weak compatibility collection of self maps. A $n \geq 3$ is arbitrary in a generalized $\mathcal{M}$-fuzzy metric space, we establish a fixed point theorem for an arbitrary collection of self maps. Finally we define multifunction and study a fixed point theorem for some multifunction on a generalized $\mathcal{M}$-fuzzy metric space.

1. Preliminaries
**Definition 1.1.** A binary operation \(*: [0,1] \times [0,1] \rightarrow [0,1]\) is a continuous \(t\)-norm if it satisfies the following conditions

(i) \(*\) is associative and commutative
(ii) \(a \ast 1 = a\) for all \(a \in [0,1]\)
(iii) \(a \ast b \leq c \ast d\) whenever \(a \leq c\) and \(b \leq d\), for each \(a,b,c,d \in [0,1]\)
(iv) \(*\) is continuous.

**Example 1.2.** The following are some usual examples of continuous \(t\)-norm

\[ a \ast b = ab \]  the product of \(a\) and \(b\),  
\[ a \ast b = \min\{a,b\}. \]

**Definition 1.3** [20]. A 3-tuple \((X, M, \ast)\) is called a \(M\)-fuzzy metric space if \(X\) is an arbitrary nonempty set, \(*\) is a continuous \(t\)-norm, and \(M\) is a fuzzy set on \(X^3 \times (0,\infty)\), satisfying the following conditions: for each \(x,y,z,a \in X\) and \(t,s > 0\).

(i) \(M(x,y,z,t) > 0\),  
(ii) \(M(x,y,z,t) = 1\) if and only if \(x = y = z\)
(iii) \(M(x,y,z,t) = M(p\{x,y,z\},t)\) (symmetry) where \(p\) is a permutation function  
(iv) \(M(x,y,a,t) \ast M(a,z,s) \leq M(x,y,z,t+s)\),  
(v) \(M(x,y,z,\cdot):(0,\infty) \rightarrow [0,1]\) is continuous

**Definition 1.4** [21]. A 3-tuple \((X, M, \ast)\) is called a generalized \(M\)-fuzzy metric space if \(X\) is an arbitrary nonempty set, \(*\) is a continuous \(t\)-norm and \(M:X^n \times (0,\infty) \rightarrow [0,1]\), \(n \geq 3\) satisfying the following conditions: for each \(x_1, x_2, \ldots, x_n, x'_n \in X\) and \(t,s > 0\)

(i) \(M(x_1, x_2, \ldots, x_n, t) > 0\),  
(ii) \(M(x_1, x_2, \ldots, x_n, t) = 1\) for all \(t > 0\) if and only if \(x_1 = x_2 = \ldots = x_n\)
(iii) \(M(x_1, x_2, \ldots, x_n, t) = M(p\{x_1, x_2, \ldots, x_n\},t)\) where \(p\) is a permutation function  
(iv) \(M(x_1, x_2, \ldots, x_n, t+s) \geq M(x_1, x_2, \ldots, x'_n, t) \ast M(x'_n, x_n, \ldots, x_n, s)\)  
(v) \(M(x_1, x_2, \ldots, x_n, \cdot):(0,\infty) \rightarrow [0,1]\) is continuous  
(vi) \(M(x_1, x_2, \ldots, x_n, t) \rightarrow 1\) as \(t \rightarrow \infty\).

**Example 1.5.** Let \(*\) be defined by \(a \ast b = ab\) or \(a \ast b = a \wedge b\). Let \((X, M, \ast)\) be a \(KM\) fuzzy metric space as defined in [15] and \(M(x_1, x_2, \ldots, x_n, t) = M(x_1, x_2, t) \ast M(x_2,\ldots,x_n, t)\).
Some Fixed Point Theorem on Generalized $\mathcal{M}$-fuzzy Metric Space

$x_i, t) \cdots M(x_n, x, t)$ for every $x_1, x_2, \ldots, x_n \in X$. Then $(X, \mathcal{M}, \ast)$ is a generalized $\mathcal{M}$-fuzzy metric space.

Remark 1.6. It is easy to observe that in a generalized $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, \ast)$, $\mathcal{M}(x, x, x, \ldots, y, t) = \mathcal{M}(y, x, x, \ldots, x, t)$ for all $t > 0$ and $x, y \in X$.

Definition 1.7. Let $(X, \mathcal{M}, \ast)$ be a generalized $\mathcal{M}$-fuzzy metric space. A sequence $\{x_k\}$ in $X$ is said to converge to a point $x \in X$ if and only if $\mathcal{M}(x_k, x, x, \ldots, x, t) \to 1$ as $k \to \infty$ for all $t > 0$.

Definition 1.8. Let $(X, \mathcal{M}, \ast)$ be a generalized $\mathcal{M}$-fuzzy metric space. A sequence $\{x_k\}$ in $X$ is said to be Cauchy sequence if $\mathcal{M}(x_k, x_m, \ldots, x_m, t) \to 1$ as $k, m \to \infty$ for all $t > 0$.

Lemma 1.9. Let $(X, \mathcal{M}, \ast)$ be a generalized $\mathcal{M}$-fuzzy metric space. Then $\mathcal{M}$ is a continuous function on $X^* \times (0, \infty)$.

Definition 1.10. A generalized $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, \ast)$ is said to be complete, if every Cauchy sequence converges.

2. Fixed points for self maps

Definition 2.1. Let $(X, \mathcal{M}, \ast)$ be a generalized $\mathcal{M}$-fuzzy metric space. Let $T_1$ and $T_2$ be self maps on $X$. If there exists a sequence $\{x_k\}$ in $X$ and $x \in X$ such that $\lim_{k \to \infty} \mathcal{M}(T_1 x_k, x, x, \ldots, x, t) = 1$ and $\lim_{k \to \infty} \mathcal{M}(T_2 x_k, x, x, \ldots, x, t) = 1$. Then $T_1$ and $T_2$ are said to have the same equivalent property.

Definition 2.2. Let $(X, \mathcal{M}, \ast)$ be a generalized $\mathcal{M}$-fuzzy metric space. Let $T_1$ and $T_2$ be self maps on $X$.

(i) The pair $T_1$ and $T_2$ of self maps on $X$ is said to be weakly compatible, if they commute at their coinciding point. I.e., $T_1 x = T_2 x$ implies $T_1 T_2 x = T_2 T_1 x$.

(ii) The pair $T_1$ and $T_2$ of self maps on $X$ is said to be compatible, if $\lim_{k \to \infty} \mathcal{M}(T_1 T_2 x_k, T_2 T_1 x_k, x, x, \ldots, x, t) = 1$, for all $t > 0$ whenever $\{x_k\}$ is a sequence in $X$ such that $\lim_{k \to \infty} T_1 x_k = \lim_{k \to \infty} T_2 x_k = x \in X$.

Definition 2.3. Let $(X, \mathcal{M}, \ast)$ be a generalized $\mathcal{M}$-fuzzy metric space. Let $T_1, \ldots, T_k$ be self maps on $X$. They are said to be weakly compatible, if any pair of $T_1, \ldots, T_k$ is weakly compatible. They are said to be compatible, if any pair of $T_1, \ldots, T_k$ is compatible.
They are said to have the same equivalent property if there exists a sequence \( \{ x_k \} \) in \( X \) and \( x \in X \) such that \( \lim_{k \to \infty} M(T_1 x_1, x, \cdots, x, t) = \cdots = \lim_{k \to \infty} M(T_n x_1, x, \cdots, x, t) = 1 \).

**Theorem 2.4.** Any pair \( T_1 \) and \( T_2 \) of compatible maps is weakly compatible.

**Proof.** Let \( T_1 x = T_2 x \) for some \( x \in X \). Consider \( \{ x_k \} \), where \( x_k = x \) for all \( k \).

Then \( \lim_{k \to \infty} T_1 x_k = T_1 x \) and \( \lim_{k \to \infty} T_2 x_k = T_2 x \). Hence \( \lim_{k \to \infty} T_1 x_k = \lim_{k \to \infty} T_2 x_k = T_1 x = T_2 x \) and so by definition of compatible maps \( \lim_{k \to \infty} M(T_1 T_2 x_k, T_2 T_1 x_k, \cdots, T_2 T_1 x_k, t) = 1 \), for all \( t > 0 \). This means that \( M(T_1 T_2 x, T_2 T_1 x, \cdots, T_2 T_1 x, t) = 1 \) for all \( t > 0 \). Hence \( T_1 T_2 x = T_2 T_1 x \).

**Theorem 2.5.** Let \((X, M, \ast)\) be a generalized \( M \)-fuzzy metric space. Then any compatible collection is weakly compatible.

**Theorem 2.6.** Let \((X, M, \ast)\) be a generalized \( M \)-fuzzy metric space. A collection of weakly compatible maps need not be compatible.

**Proof.** Consider \( X = [0, 2] \) and \( \ast \) is given by \( a \ast b = ab \), the usual product of \( a \) and \( b \). Define \( M(x_1, x_2, \cdots, x_n, t) = e^{\frac{-\|x_1-x_2\|}{t}} \times \cdots \times e^{\frac{-\|x_{n-1}-x_n\|}{t}} \). Then \((X, M, \ast)\) is a generalized \( M \)-fuzzy metric space. Consider the maps \( T_1, T_2, T_3 \) defined as follows

\[
T_1(x) = \begin{cases} 
2 - x & \text{if } 0 \leq x < 1 \\
x & \text{if } 1 \leq x \leq 2 \\
2 & \text{if } 1 \leq x \leq 2 
\end{cases},
T_2(x) = \begin{cases} 
(3/2) - (x/2) & \text{if } 0 \leq x < 1 \\
2 & \text{if } 1 \leq x \leq 2 
\end{cases},
T_3(x) = \begin{cases} 
(3/4) + (x/4) & \text{if } 0 \leq x < 1 \\
2 & \text{if } 1 \leq x \leq 2 
\end{cases}.
\]

Then \( T_1 T_2(x) = 2 \) if \( 1 \leq x \leq 2 \), for all \( i, j \) and so \( T_1, T_2, T_3 \) are weakly compatible maps. Consider \( x_k = 1 - 1/k \). Then it is clear that \( \lim_{k \to \infty} M(T_1 x_k, T_2 x_k, 1, t) = 1 \), \( \lim_{k \to \infty} M(T_2 x_k, 1, T_2 x_k, t) = 1 \) and \( \lim_{k \to \infty} M(T_3 x_k, T_3 x_k, 1, t) = 1 \). Hence \( \lim_{k \to \infty} T_1 x_k = \lim_{k \to \infty} T_2 x_k = \lim_{k \to \infty} T_3 x_k = 1 \). Now \( T_1 T_2 x_k = 1 + 1/4k \), \( T_2 T_3 x_k = 2 \), \( T_2 T_4 x_k = 2 \), \( T_2 T_5 x_k = 1 + 1/8k \) and \( T_2 T_6 x_k = 2 \). Hence \( \lim_{k \to \infty} M(T_1 T_2 x_k, T_2 T_1 x_k, T_2 T_1 x_k, T_2 T_1 x_k, t) = \lim_{k \to \infty} M(T_2 T_3 x_k, T_2 T_4 x_k, \cdots, T_2 T_5 x_k, T_2 T_6 x_k, t) = \lim_{k \to \infty} e^{\frac{-1}{4k}} \times \lim_{k \to \infty} e^{\frac{-1}{8k}} \times \lim_{k \to \infty} e^{\frac{-1}{8k}} = \lim_{k \to \infty} e^{-1/16k} \times \lim_{k \to \infty} e^{-1/16k} = \lim_{k \to \infty} e^{-1/8k} = e^{-1/8} \). That is \( \lim_{k \to \infty} M(T_1 T_2 x_k, T_2 T_1 x_k, T_2 T_1 x_k, T_2 T_1 x_k, t) = 1 \). Also \( \lim_{k \to \infty} M(T_1 T_2 x_k, T_2 T_1 x_k, T_2 T_1 x_k, T_2 T_1 x_k, t) \neq 1 \). This shows that weakly compatible maps need not be compatible.

**Theorem 2.7.** Let \((X, M, \ast)\) be a generalized \( M \)-fuzzy metric space. Let \( A_1, A_2, \cdots, A_n, \ B, \ T_1, \cdots, T_n, \ S \) are self maps on \( X \) satisfying the following properties:
(i) $T_i y_1 = T_i y_2 = \cdots = T_i y_n$ implies $y_1 = y_2 = \cdots = y_n$

(ii) $B, T_1, \ldots, T_n$ have the same equivalent property

(iii) $A_i (X) \subset T_i (X)$ for $i = 1, 2, \ldots, n$ and $B (X) \subset S (X)$ and $S (X)$ is a complete space.

(iv) The collection of maps $A_1, A_2, \ldots, A_n, S$ is weakly compatible.

(v) The collection of maps $T_1, \ldots, T_n, B$ is weakly compatible.

(vi) For all $y_1, y_2, \ldots, y_{n-1}$ and $x \in X$, $M (A_1 x, B y_1, B y_2, \ldots, B y_{n-1}, t) \geq 1$ implies that $y_1 = y_2 = \cdots = y_n$.
\[ M(u, Bu, \cdots, Bu, t) = M(A_u, Bu, \cdots, Bu, t) \geq \min \{ M(S_u, T_u, \cdots, T_u, kt), \cdots, M(S_u, T_u, \cdots, T_u, kt) \}. \]

Now by (v) we have \[ M(u, Bu, \cdots, Bu, t) \geq M(S_u, Bu, \cdots, Bu, kt) = M(A_u, Bu, \cdots, Bu, kt) = M(u, Bu, \cdots, Bu, t) \]. Hence \[ M(u, Bu, \cdots, Bu, t) \geq M(u, Bu, \cdots, Bu, kt) \]. This implies that \[ u = Bu \] and so \[ Bu = T_u = u \].

**Uniqueness of** \( u \): Suppose \( u \neq v \), \( u \), \( v \) are two distinct fixed points of maps \( A_i \), \( B_i \), \( S_i \), \( i = 1, 2, \ldots, n \). Then \[ M(u, u, \cdots, u, v, t) = M(A_v, Bu, \cdots, Bu, t) \geq M(S_v, Bu, \cdots, Bu, kt) \geq \min \{ M(S_v, T_u, \cdots, T_u, kt), \cdots, M(S_v, T_u, \cdots, T_u, kt) \} \geq M(A_v, Bu, \cdots, Bu, kt) \geq M(v, u, \cdots, u, v, kt) \]. Hence \[ M(u, u, \cdots, u, v, t) \geq M(u, u, \cdots, u, v, kt) \] and so \( u = v \).

### 3. Fixed points for multivalued maps

**Definition 3.1.** Let \( (X, M, \ast) \) be a generalized \( M \)-fuzzy metric space. A multivalued map or multifunction is a mapping \( T : 2^X \rightarrow 2^X \) which assigns to each \( x \in X \), a nonempty subset \( T_x \) of \( X \). Fixed points of multifunction \( T \) are those points \( x \in X \) for which \( x \in T_x \).

**Definition 3.2.** Let \( (X, M, \ast) \) be a generalized \( M \)-fuzzy metric space. Let \( S_1, S_2, \cdots, S_n \) be subsets of \( X \) and \( x_1, x_2, \cdots, x_n \in X \). Define the following

(i) \[ M(S_1, x_2, \cdots, x_n, t) = \sup \{ M(a_1, x_2, \cdots, x_n, t) : a_1 \in S_1 \} \]

(ii) \[ M(S_1, S_2, x_3, \cdots, x_n, t) = \sup \{ M(a_1, a_2, x_3, \cdots, x_n, t) : a_1 \in S_1, a_2 \in S_2 \} \]

(iii) \[ M(S_1, S_2, \cdots, S_{n-1}, x_n, t) = \sup \{ M(a_1, a_2, \cdots, a_{n-1}, x_n, t) : a_1 \in S_1, a_2 \in S_2, \cdots, a_{n-1} \in S_{n-1} \} \]

and

(iv) \[ M(S_1, S_2, \cdots, S_n, t) = \min \left[ \inf \left\{ M(a_1, S_2, \cdots, S_n, t) \right\} \right] \]

**Definition 3.3.** Let \( (X, M, \ast) \) be a generalized \( M \)-fuzzy metric space. Let \( T_1, T_2 \) be multivalued mappings of \( X \). Then \( T_2 \subseteq T_1 \) means for any \( x_0 \in X \) with \( x_1 \in T_1 x_0 \) we have \( T_2 x_1 \subseteq T_1 x_0 \).

**Theorem 3.4.** Let \( (X, M, \ast) \) be a complete generalized \( M \)-fuzzy metric space. Let \( q \in (0, 1) \). Let \( T_1, T_2, \cdots, T_n \) be multivalued mappings with the following properties:

(i) \( T_1 \supseteq T_2 \supseteq \cdots \supseteq T_n \)
(ii) $M(T_1x_1, T_2x_2, \ldots, T_nx_n, qt) \geq \min \{M(x_1, \ldots, x_n, t), M(x_1, T_1x_1, x_1, \ldots, x_1, t), \ldots, M(x_n, T_nx_n, x_n, \ldots, x_n, t)\}$

(iii) $M(x, x, \ldots, x, t) \geq M(x, x, y, t) \geq M(x, x, y, z, t) \geq \cdots$ for all $x, y, z, \ldots \in X$

Then $T_1, T_2, \ldots, T_n$ have a same common fixed point.

Proof. Let $x_0 \in X$ and $\varepsilon > 0$ be given. Then there exists $x_1 \in T_1x_0$ such that $M(T_1x_0, x_0, \ldots, x_0, qt) - \varepsilon/2 < M(x_1, x_1, \ldots, x_1, qt)$.

Now, since $T_2x_1 \subset T_1x_0$, there exists $x_2 \in T_2x_1$ such that $M(x_1, T_2x_1, x_1, \ldots, x_1, qt) - \varepsilon/2 < M(x_1, x_2, x_1, \ldots, x_1, qt)$. Since $T_3x_2 \subset T_2x_1$, there exists $x_3 \in T_3x_2$ such that $M(x_2, T_3x_2, x_2, \ldots, x_2, qt) - \varepsilon/2 < M(x_2, x_3, x_3, \ldots, x_2, qt)$. Continuing in this way we get a sequence $x_n \in T_nx_{n-1}$ such that $M(x_n, x_{n-1}, \ldots, x_1, qt) - \varepsilon/2 < M(x_n, x_{n-1}, \ldots, x_1, qt)$.

Similarly we will get $x_n \in T_nx_{n-1}$ such that $M(T_nx_n, x_{n-1}, \ldots, x_1, qt) - \varepsilon/2 < M(x_n, x_{n-1}, \ldots, x_1, qt)$ etc. Proceeding in this way we get a sequence $\{x_n\}$.

Claim: $\{x_n\}$ is a Cauchy sequence. It is clear that $M(x_{nk+1}, x_{nk+1}, x_{nk+1}, qt) \geq M(T_nx_{nk+1}, x_{nk+1}, x_{nk+1}, qt) - \varepsilon/2^k \geq \min \{M(x_{nk}, x_{nk+1}, x_{nk+1}, t), M(x_{nk}, \ldots, x_{nk}, T_nx_{nk-1}, t)\} - \varepsilon/2^k \geq \min \{M(x_{nk}, x_{nk+1}, \ldots, x_{nk+k-1}, t), M(x_{nk}, x_{nk+1}, t), \ldots, M(x_{nk+k-1}, x_{nk+1}, x_{nk+1}, t)\} - \varepsilon/2^k \geq M(x_{nk}, x_{nk+1}, \ldots, x_{nk+k-1}, t) - \varepsilon/2^k$.

Now, $M(T_nx_{nk+1}, x_{nk+1}, x_{nk+1}, t) \geq M(T_nx_{nk+1}, T_nx_{nk+1}, x_{nk+1}, t) \geq M(T_nx_{nk+1}, \ldots, x_{nk+1}, t) \geq \min \{M(x_{nk}, x_{nk+1}, \ldots, x_{nk+k-1}, t/q), M(x_{nk}, x_{nk+1}, x_{nk+1}, t/q), \ldots, M(x_{nk+k-1}, \ldots, x_{nk+1}, t/q)\}$.

Hence $M(x_{nk}, x_{nk+1}, \ldots, x_{nk+k-1}, t/q) \geq M(x_{nk}, x_{nk+1}, \ldots, x_{nk+k-1}, x_{nk+k-1}, t/q) \geq M(x_{nk}, x_{nk+1}, \ldots, x_{nk+k-1}, t/q) \geq M(x_{nk}, x_{nk+1}, \ldots, x_{nk+k-1}, t/q) \geq \cdots \geq M(x_{nk}, x_{nk+1}, \ldots, x_{nk+k-1}, t/q)$ etc. This means that $M(x_{nk}, x_{nk+1}, \ldots, x_{nk+k-1}, t/q) \to 1$ as $k \to \infty$. This implies that $M(x_{nk}, x_{nk+1}, \ldots, x_{nk+k-1}, t/q) \to 1$ as $k \to \infty$.
\[
\left\{ M\left(x_{nk-1}, x_{nk-2}, \ldots, x_{nk-1}t\right), M\left(x_{nk-1}, \ldots, x_{nk-1}t\right), M\left(x_{nk-1}, x_{nk-2}, x_{nk-1}t\right), \ldots, M\left(x_{nk-1}, x_{nk-1}t, x_{nk-1}t\right) \right\} = \min \left\{ M\left(x_{nk-1}, x_{nk-1}t\right), M\left(x_{nk-1}, x_{nk-1}t\right), M\left(x_{nk-1}, x_{nk-1}t\right), \ldots, M\left(x_{nk-1}, x_{nk-1}t, x_{nk-1}t\right) \right\} \]

\[
M\left(x_{nk-1}, x_{nk-1}t, x_{nk-1}t\right) = M\left(x_{nk-1}, x_{nk-1}t, x_{nk-1}t\right) = M\left(x_{nk-1}, x_{nk-1}t, x_{nk-1}t\right). \]  

Hence \( M\left(x_{nk-1}, x_{nk-1}t, x_{nk-1}t\right) \to 1 \) as \( k \to \infty \). This implies that \( M\left(x_{nk-1}, x_{nk-1}t, x_{nk-1}t\right) \to 1 \) as \( k \to \infty \). Now for any \( k \), \( m \) with \( m > k \) we have \( M\left(x_{nk-1}, x_{nk-1}t, x_{nk-1}t\right) \to M\left(x_{nk-1}, x_{nk-1}t, x_{nk-1}t\right) \).  

Since \( X, M, \ast \) is a generalized \( M \)-fuzzy metric space, there is a \( z \) in \( X \) such that \( x_{nk-1} \to z \). Next to prove \( z \in T_z \) for all \( i = 1, 2, \ldots, n \). 

It is clear that \( M\left(T_z, x_{nk-1}, x_{nk-1}t\right) \to M\left(T_z, x_{nk-1}, x_{nk-1}t\right) \). Hence \( M\left(T_z, x_{nk-1}, x_{nk-1}t\right) \to M\left(T_z, x_{nk-1}, x_{nk-1}t\right) \) as \( k \to \infty \). Since \( X, M, \ast \) is a generalized \( M \)-fuzzy metric space, there is a \( z \) in \( X \) such that \( x_{nk-1} \to z \). Next to prove \( z \in T_z \) for all \( i = 1, 2, \ldots, n \). 

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Some Fixed Point Theorem on Generalized $\mathcal{M}$-fuzzy Metric Space


Fuzzy Implication Groupoids

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Abstract:
In this paper, we fuzzify the concept of implication groupoids and investigate some of their properties. We give a characterization of fuzzy implication groupoid, and discuss a characterization of fuzzy implication groupoids in terms of level subalgebras of fuzzy implication groupoids.

Keywords:
Implication groupoid, distributive implication groupoid, level subalgebra, normal fuzzy implication groupoid.

1. Introduction

I. Chajda and R. Halas [4] introduced the concept of implication groupoid as a generalization of the implication reduct of intuitionistic logic, i.e. a Hilbert algebra and studied some connections among ideals, deductive systems and congruence kernels whenever implication groupoid is distributive. In [2], R. K. Bandaru further studied the properties of ideals in implication groupoids. In [6,7], Y. B. Jun et. al introduced the concept of fuzzy ideal, fuzzy deductive systems in Hilbert algebras and discuss the relation between fuzzy ideals and fuzzy deductive systems. In this paper we fuzzify the concept of implication groupoids and investigate some of their properties. We give a characterization of fuzzy implication groupoid, and discuss a characterization of fuzzy implication groupoids in terms of level subalgebras of fuzzy implication groupoids.

2. Preliminaries
Definition 2.1 [4]. An algebra \((A,\ast,1)\) of type \((2,0)\) is called an implication groupoid if it satisfies the identities:

1. \(x \ast x = 1\)
2. \(1 \ast x = x\) for all \(x, y \in A\).

Definition 2.2 [4]. An implication groupoid \((A,\ast,1)\) of type \((2,0)\) is called a distributive implication groupoid if it satisfies the following identity:

\[(LD) \quad x \ast (y \ast z) = (x \ast y) \ast (x \ast z) \quad (\text{left distributivity})\]

for all \(x, y, z \in A\).

Example 2.3. Let \(A = \{1, a, b, c, d\}\) in which \(\ast\) is defined by

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Then \((A,\ast,1)\) is a distributive implication groupoid.

In every implication groupoid, one can introduce the so-called induced relation \(\leq\) by the setting \(x \leq y\) if and only if \(x \ast y = 1\); the induced relation \(\leq\) is a quasiorder.

Lemma 2.4 [4]. Let \((A,\ast,1)\) be a distributive implication groupoid. Then \(A\) satisfies the identities

\(x \ast 1 = 1 \quad \text{and} \quad x \ast (y \ast x) = 1\)

3. Fuzzy implication groupoids

Throughout this section, implication groupoid \((X,\ast,1)\) means distributive implication groupoid. First we begin with the following definition.

Definition 3.1 [7]. Let \(X\) be a set. A fuzzy set in \(X\) is a function \(\mu : X \to [0,1]\).

Definition 3.2 [7]. Let \(\mu\) be a fuzzy set in a set \(X\). For \(\alpha \in [0,1]\), the set \(\mu_{\alpha} = \{x \in X | \mu(x) \geq \alpha\}\) is called a level subset of \(\mu\).

Definition 3.3. A fuzzy set \(\mu\) in \(X\) is called a fuzzy implication groupoid of \(X\) if it satisfies, for all \(x, y \in X\), \(\mu(x \ast y) \geq \min\{\mu(x), \mu(y)\}\).
Example 3.4. Let \( X = \{1, a, b, c, d\} \) be the implication groupoid an in Example 2.3 and \( A = \{1, a, b\} \). Let \( t_1, t_2 \in [0,1] \) be such that \( t_1 > t_2 \). Define a mapping \( \mu : X \to [0,1] \) by \( \mu (1) = \mu (a) = \mu (b) = t_1 \) and \( \mu (c) = \mu (d) = t_2 \). Then \( \mu \) is a fuzzy implication groupoid of \( X \).

Corollary 3.5. If \( \mu \) is a fuzzy implication groupoid of \( X \), then \( \mu (1) \geq \mu (x) \) for all \( x \in X \).

Proposition 3.6. Let \( \mu \) be a fuzzy implication groupoid of \( X \) and \( a \in X \). If \( \mu \) is decreasing, then it is constant.

Proof. We note that \( \mu (x) \leq \mu (1) \) for all \( x \in X \). Since \( x \leq 1 \) for all \( x \in X \), \( \mu (x) \geq \mu (1) \) because \( \mu \) is decreasing. Hence \( \mu (x) = \mu (1) \) for all \( x \in X \). Thus \( \mu \) is constant.

Theorem 3.7. Let \( \mu \) be a fuzzy set in implication groupoid \( X \). Then \( \mu \) is a fuzzy implication groupoid of \( X \) if and only if for every \( \alpha \in [0,1] \), the level subset \( \mu_{\alpha} \) is a subalgebra of \( X \), when \( \mu_{\alpha} \neq \emptyset \).

Proof. Let \( \mu \) be a fuzzy implication groupoid of \( X \) and \( x, y \in \mu_{\alpha} \) for every \( \alpha \in [0,1] \) with \( \mu_{\alpha} \neq \emptyset \). Then \( \mu(x \circledast y) \geq \min \{\mu(x), \mu(y)\} \geq \alpha \), which implies that \( \mu(x \circledast y) \geq \alpha \). Hence \( x \circledast y \in \mu_{\alpha} \). Thus \( \mu_{\alpha} \) is a subalgebra of \( X \). Conversely, assume that \( \mu_{\alpha} \) is a subalgebra of \( X \) for every \( \alpha \in [0,1] \) with \( \mu_{\alpha} \neq \emptyset \). Let \( x, y \in X \) and \( \mu(x) = \alpha_1 \) and \( \mu(y) = \alpha_2 \). Then \( x \in \mu_{\alpha_1} \) and \( y \in \mu_{\alpha_2} \). Without loss of generality, we may assume that \( \alpha_1 \leq \alpha_2 \). Then \( \mu_{\alpha_1} \subseteq \mu_{\alpha_2} \) and so \( y \in \mu_{\alpha_1} \). Since \( \mu_{\alpha_1} \) is a subalgebra of \( X \), we have \( x \circledast y \in \mu_{\alpha_1} \). Hence \( \mu(x \circledast y) \geq \alpha_1 = \min \{\mu(x), \mu(y)\} \). Therefore \( \mu \) is a fuzzy implication groupoid of \( X \).

Corollary 3.8. If \( \mu \) is a fuzzy implication groupoid of \( X \), then the set \( X_\mu = \{x \in X | \mu(x) = \mu(1)\} \) is a subalgebra of \( X \).

Proof. Since \( \mu(x) \leq \mu(1) \) for all \( x \in X \), we have

\[
\mu_{\mu(1)} = \{x \in X | \mu(x) \geq \mu(1)\} = \{x \in X | \mu(x) = \mu(1)\} = X_\mu.
\]

By Theorem 3.7, we know that \( X_\mu \) is a subalgebra of \( X \).

Definition 3.9. Let \( \mu \) be a fuzzy implication groupoid of \( X \). Each subalgebra \( \mu_{\alpha} \) of \( X \), \( \alpha \in [0,1] \), is called a level subalgebra of \( \mu \), when \( \mu_{\alpha} \neq \emptyset \).
Theorem 3.10. Let $A$ be a subalgebra of implication groupoid $X$ and let $\mu : X \rightarrow [0, 1]$ be a fuzzy set defined by, for all $x \in X$,

$$\mu(x) = \begin{cases} 
\alpha_0 & \text{if } x \in A \\
\alpha_1 & \text{if } x \notin A
\end{cases}$$

where $\alpha_0, \alpha_1 \in [0, 1]$ and $\alpha_0 > \alpha_1$. Then $\mu$ is a fuzzy implication groupoid of $X$.

Proof. Let $x, y \in X$. If at least one of $x$ and $y$ does not belong to $A$, then $\mu(x \ast y) \geq \alpha_i = \min \{\mu(x), \mu(y)\}$, since $\alpha_i$ is the minimum value of $\mu$. If $x, y \in A$, then $x \ast y \in A$. Hence $\mu(x \ast y) = \min \{\mu(x), \mu(y)\}$. Therefore $\mu$ is a fuzzy implication groupoid of $X$.

Corollary 3.11. Any proper subalgebra of implication groupoid of $X$ can be realized as a level subalgebra of some fuzzy implication groupoid of $X$.

Proof. Let $A$ be a proper subalgebra of $X$ and let $\mu$ be a fuzzy set in $X$ defined by

$$\mu(x) = \begin{cases} 
\alpha & \text{if } x \in A \\
0 & \text{if } x \notin A.
\end{cases}$$

where $\alpha$ is fixed number in $(0, 1]$. Taking $\alpha_0 = \alpha$ and $\alpha_1 = 0$ in Theorem 3.10, we know that $\mu$ is a fuzzy implication groupoid of $X$, and obviously $\mu_A = A$. This completes the proof.

We can generalize Theorem 3.10 as follows:

Theorem 3.12. Let $\{A_n\}_{n=0}^\infty$ be a strictly decreasing sequence of subalgebras of implication groupoid $X = A_0$ and let $\{\alpha_n\}_{n=0}^\infty$ be a strictly increasing sequence in $(0, 1)$. Then there is a fuzzy implication groupoid $\mu$ of $X$ such that $\mu_{\alpha_n} = A_n$ for all $n = 0, 1, 2, \cdots$.

Proof. Define a fuzzy set $\mu : X \rightarrow [0, 1]$ by

$$\mu(x) = \begin{cases} 
\alpha_n & \text{if } x \in A_n - A_{n+1} \\
\lim_{n \to \infty} \alpha_n & \text{if } x \in \bigcap_{n=1}^\infty A_n.
\end{cases}$$

It can be easily seen that $\mu$ is a fuzzy implication groupoid of $X$ and that $\mu_{\alpha_n} = A_n$ for all $n = 0, 1, 2, \cdots$. 
**Theorem 3.13.** Let $\mu$ be a fuzzy implication groupoid of $X$. Then two level subalgebras $\mu_{\alpha_i}, \mu_{\alpha_j}$ with $\alpha_i < \alpha_j$ of $\mu$ are equal if and only if there is no $x \in X$ such that $\alpha_i \leq \mu(x) < \alpha_j$.

**Proof.** Suppose that $\alpha_i < \alpha_j$ and $\mu_{\alpha_i} = \mu_{\alpha_j}$. If there exists $x \in X$ such that $\alpha_i \leq \mu(x) < \alpha_j$, then $\mu_{\alpha_i}$ is a proper subset of $\mu_{\alpha_j}$. This is impossible. Conversely, suppose that there is no $x \in X$ such that $\alpha_i \leq \mu(x) < \alpha_j$. Note that $\alpha_i < \alpha_j$ implies $\mu_{\alpha_i} \subseteq \mu_{\alpha_j}$. If $x \in \mu_{\alpha_i}$, then $\mu(x) \geq \alpha_i$, and so $\mu(x) \geq \alpha_j$ because $\mu(x) \in \alpha_j$. Hence $x \in \mu_{\alpha_j}$, which says that $\mu_{\alpha_i} \subseteq \mu_{\alpha_j}$. Thus $\mu_{\alpha_i} = \mu_{\alpha_j}$. This completes the proof.

**Remark 3.14.** As a consequence of Theorem 3.13, the level subalgebras of a fuzzy implication groupoid $\mu$ of $X$ which has a countable image form a chain. But $\mu(1)$ for all $x \in X$, and so $\mu_{\alpha(i)}$ is the smallest level subalgebra of a fuzzy implication groupoid, but not always $\mu_{\alpha(i)} = \{1\}$ as shown in the following example. Thus we have a chain $X = \mu_{\alpha_0} \supseteq \mu_{\alpha_1} \supseteq \cdots \supseteq \mu_{\alpha_n} \supseteq \cdots$, where $\alpha_0 < \alpha_1 < \cdots < \alpha_n < \cdots$ and $\mu(1) = \lim_{n \to \infty} \alpha_n$.

**Example 3.15.** Let $A$ be a proper subalgebra of implication groupoid $X$ and let $\mu$ be a fuzzy implication groupoid of $X$ in the proof to Corollary 3.11. Then $\text{Im}(\mu) = \{0, \alpha\}$, and two level subalgebras of $\mu$ are $\mu_0 = X$ and $\mu_n = A$. Thus we have $\mu(1) = \alpha$ but $\mu_\alpha = A \neq \{1\}$.

**Corollary 3.16.** Let $\mu$ be a fuzzy implication groupoid of $X$. If $\text{Im}(\mu) = \{\alpha_1, \cdots, \alpha_n\}$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, then the family of subalgebras $\mu_{\alpha_i}$ of $\mu(i=1,2,\cdots,n)$ constitutes all the level subalgebras of $\mu$.

**Proof.** Let $\alpha \in [0,1]$ and $\alpha \not\in \text{Im}(\mu)$. If $\alpha < \alpha_1$, then $\mu_{\alpha_1} \subseteq \mu_{\alpha}$. Since $\mu_{\alpha_1} = X$, we have $\mu_{\alpha} = X$ and $\mu_{\alpha} = \mu_{\alpha_1}$. If $\alpha_i < \alpha < \alpha_{i+1}$ ($1 \leq i \leq n-1$), then there is no $x \in X$ such that $\alpha \leq \mu(x) < \alpha_{i+1}$. Using Theorem 3.13, we obtain $\mu_{\alpha} = \mu_{\alpha_{i+1}}$. This shows that for any $\alpha \in [0,1]$ with $\alpha \leq \mu(1)$, the level subalgebra $\mu_{\alpha}$ is in $\{\mu_{\alpha_i} \mid i \leq n\}$.

**Theorem 3.17.** Let $\mu$ be a fuzzy implication groupoid of $X$ with $\text{Im}(\mu) = [\alpha_i, \beta_i]$, where $\Delta$ is an arbitrary index set. Then

(i) There exists a unique $\beta_i \in \Delta$ such that $\alpha_i > \beta_i$ for all $i \in \Delta$.

(ii) $X_{\mu} = \bigcap_{i \in \Delta} \mu_{\beta_i} = \mu_{\alpha_i}$.

(iii) $X = \bigcup_{i \in \Delta} \mu_{\beta_i}$.

(iv) The members of $A$ form a chain.
(v) If $\mu$ attains its infimum on all subalgebras of $X$, then $A$ contains all level subalgebras of $\mu$.

**Proof.**

(i) Since $\mu(1) \in \text{Im}(\mu)$, there exists a unique $i_0 \in \Delta$ such that $\alpha_{i_0} = \mu(1) \geq \mu(x)$ for all $x \in X$ so that $\alpha_{i_0} \geq \alpha_i$ for all $i \in \Delta$.

(ii) We know that $\mu_{i_0} = \{x \in X | \mu(x) \geq \alpha_{i_0} \} = \{x \in X | \mu(x) = \alpha_{i_0} \} = \{x \in X | \mu(x) = \mu(1) \} = X_{i_0}$. Since $\alpha_{i_0} \geq \alpha_i$ for all $i \in \Delta$, therefore clearly $\mu_{i_0} \subseteq \mu_i$ for all $i \in \Delta$.

Hence $\mu_{i_0} \subseteq \bigcap_{i \in \Delta} \mu_i$, and so $\mu_{i_0} = \bigcap_{i \in \Delta} \mu_i$, because $i_0 \in \Delta$.

(iii) It is sufficient to show that $X \subseteq \bigcup_{i \in \Delta} \mu_i$. Let $x \in X$. Then $\mu(x) \in \text{Im}(\mu)$ and so there exists $i(x) \in \Delta$ such that $\mu(x) = \alpha_{i(x)}$. This implies $x \in \mu_{i(x)} \subseteq \bigcup_{i \in \Delta} \mu_i$. This proves (iii).

(iv) Note that for any $i, j \in \Delta$, either $\alpha_i \geq \alpha_j$ or $\alpha_i \leq \alpha_j$, hence $\mu_{i} \subseteq \mu_{i_0}$ or $\mu_{i_0} \subseteq \mu_{i_0}$. Therefore the members of $A$ form a chain.

(v) Assume that $\mu$ attains its infimum on all subalgebras of $X$. Let $\mu_{i_0}$ be a level subalgebra of $\mu$. If $\alpha = \alpha_{i_0}$ for some $i \in \Delta$, then clearly $\mu_{i_0} \subseteq A$. Assume that $\alpha \neq \alpha_{i_0}$ for all $i \in \Delta$. Then there is no $x \in X$ such that $\mu(x) = \alpha$. Let $A = \{x \in X \mid \mu(x) > \alpha \}$. Obviously $1 \in A$, and so $A \neq \emptyset$. Let $x, y \in A$. Then $\mu(x) > \alpha$ and $\mu(y) > \alpha$. Since $\mu$ is a fuzzy implication groupoid of $X$, it follows that $\mu(x*y) \geq \min \{\mu(x), \mu(y)\} > \alpha$ so that $\mu(x*y) > \alpha$, i.e., $x*y \in A$. Hence $A$ is a subalgebra of $X$. By hypothesis, there exists $y \in A$ such that $\mu(y) = \inf \{\mu(x) | x \in X\}$. Now $\mu(y) \in \text{Im}(\mu)$ implies $\mu(y) = \alpha_i$ for some $i \in \Delta$. Obviously $\alpha_i \geq \alpha$, and so by assumption $\alpha_i > \alpha$. Note that there is no $z \in X$ such that $\alpha \leq \mu(z) < \alpha_i$. It follows from Theorem 3.13 that $\mu_{i_0} = \mu_{i_0}$. Hence $\mu_{i_0} \in A$. This completes the proof.

4. Normal fuzzy implication groupoids

**Definition 4.1.** A fuzzy implication groupoid $\mu$ of $X$ is said to be normal if there exists $x \in X$ such that $\mu(x) = 1$.

Let $\mu$ and $\sigma$ be any two fuzzy subsets of a set $X$. Then $\mu$ is said to be contained in $\sigma$, denoted by $\mu \subseteq \sigma$, if $\mu(x) \leq \sigma(x)$ for all $x \in X$. If $\mu(x) = \sigma(x)$ for all $x \in X$, $\mu$ and $\sigma$ are said to be equal and we write $\mu = \sigma$. We note that if $\mu$ is a normal fuzzy implication groupoid of $X$, then $\mu(1) = 1$. Hence we have the following characterization.

**Theorem 4.2.** A fuzzy implication groupoid $\mu$ of $X$ is normal if and only if $\mu(1) = 1$. 

Theorem 4.3. If $\mu$ is a fuzzy implication groupoid of $X$, then the fuzzy set $\mu^+$ of $X$ defined by $\mu^+(x) = \mu(x) + 1 - \mu(1)$ for all $x \in X$ is a normal fuzzy implication groupoid of $X$ containing $\mu$.

Proof. Assume that $\mu$ is a fuzzy implication groupoid of $X$ and let $x, y \in X$. Then $\mu^+(x \ast y) = \mu(x \ast y) + 1 - \mu(1) \geq \min\{\mu(x), \mu(y)\} + 1 - \mu(1) = \min\{\mu(x) + 1 - \mu(1), \mu(y) + 1 - \mu(1)\} = \min\{\mu^+(x), \mu^+(y)\}$ and $\mu^+(1) = \mu(1) + 1 - \mu(1) = 1$. Hence $\mu^+$ is a normal fuzzy implication groupoid of $X$, and clearly $\mu \subseteq \mu^+$.

Theorem 4.4. Let $\mu$ and $\nu$ be fuzzy implication groupoids of $X$. If $\mu \subseteq \nu$ and $\mu(1) = \nu(1)$, then $X_{\mu} \subseteq X_{\nu}$.

Proof. Assume that $\mu \subseteq \nu$ and $\mu(1) = \nu(1)$. If $x \in X_{\mu}$, then $\nu(x) \geq \mu(x) = \mu(1) = \nu(1)$. Noticing that $\nu(x) \leq \nu(1)$ for all $x \in X$, we have $\nu(x) = \nu(1)$, i.e., $x \in X_{\nu}$.

Corollary 4.5. If $\mu$ and $\nu$ are normal fuzzy implication groupoids of $X$ satisfying $\mu \subseteq \nu$, then $X_{\mu} \subseteq X_{\nu}$.

Theorem 4.6. A fuzzy implication groupoid $\mu$ of $X$ is normal if and only if $\mu^+ = \mu$.

Proof. The sufficiency is obvious. Assume that $\mu$ is a normal fuzzy implication groupoid of $X$ and let $x \in X$. Then $\mu^+(x) = \mu(x) + 1 - \mu(1) = \mu(x)$, and hence $\mu^+ = \mu$.

Theorem 4.7. If $\mu$ is a fuzzy implication groupoid of $X$, then $(\mu^+)^+ = \mu^+$.

Proof. For any $x \in X$, we have $(\mu^+)^+(x) = \mu^+(x) + 1 - \mu^+(1) = \mu^+(\mu^+(x))$, completing the proof.

Theorem 4.8. Let $\mu$ be a fuzzy implication groupoid of $X$. If there exists a fuzzy implication groupoid $\nu$ of $X$ satisfying $\nu^+ \subseteq \mu$, then $\mu$ is normal.

Proof. Suppose there exists a fuzzy implication groupoid $\nu$ of $X$ such that $\nu^+ \subseteq \mu$. Then $1 = \nu^+(1) \leq \mu(1)$ and hence $\mu(1) = 1$.

Corollary 4.9. Let $\mu$ be a fuzzy implication groupoid of $X$. If there exists a fuzzy implication groupoid $\nu$ of $X$ satisfying $\nu^+ \subseteq \mu$, then $\mu^+ = \mu$.

Theorem 4.10. Let $\mu$ be a fuzzy implication groupoid of $X$ and let $f : [0, \mu(1)] \rightarrow [0, 1]$ be an increasing function. Define a fuzzy set $\mu_f : X \rightarrow [0, 1]$ by $\mu_f(x) = f(\mu(x))$. 
for all $x \in X$. Then $\mu_j$ is a fuzzy implication groupoid of $X$. In particular, if $f(\mu(1)) = 1$, then $\mu_j$ is normal, and if $f(\alpha) \geq \alpha$ for all $\alpha \in [0, \mu(1)]$, then $\mu \subset \mu_j$.

**Proof.** Let $x, y \in X$. Then $\mu_j(x*y) = f(\mu(x \mu y)) \geq f(\min \{\mu(x), \mu(y)\}) = \min \{f(\mu(x)), f(\mu(y))\} = \min \{\mu_j(x), \mu_j(y)\}$. Hence $\mu_j$ is a fuzzy implication groupoid of $X$. If $f(\mu(1)) = 1$ then clearly $\mu_j$ is normal. Assume that $f(\alpha) \geq \alpha$ for all $\alpha \in [0, \mu(1)]$. Then $\mu_j(x) = f(\mu(x)) \geq \mu(x)$ for all $x \in X$, which proves that $\mu \subset \mu_j$.

**Theorem 4.11.** Let $\mu$ be a non-constant normal fuzzy implication groupoid of $X$, which is maximal in the poset of normal fuzzy implication groupoids under set inclusion. Then $\mu$ takes only the values 0 and 1.

**Proof.** By Theorem 4.2, $\mu(1) = 1$. Let $x \in X$ be such that $\mu(x) \neq 1$. It sufficient to show that $\mu(x) = 0$. Assume that there exists $a \in X$ such that $0 < \mu(a) < 1$. Define a fuzzy set $\nu : X \to [0, 1]$ by $\nu(x) = \frac{1}{2}\{\mu(x) + \mu(a)\}$ for all $x \in X$. Then clearly $\nu$ is well-defined. Let $x, y \in X$. Then $\nu(x*y) = \frac{1}{2}\{\mu(x*y) + \mu(\alpha)\} = \frac{1}{2}\mu(x*y) + \frac{1}{2}\mu(a) \geq \frac{1}{2}\min\{\mu(x); \mu(y)\} + \frac{1}{2}\mu(a) = \frac{1}{2}\{\mu(x) + \mu(a)\} + \frac{1}{2}\mu(a) = \frac{1}{2}\{\nu(x) + \nu(y)\}$, which proves that $\nu$ is a fuzzy implication groupoid of $X$. Now we have $\nu^+(x) = \nu(x) + 1 - \nu(1) = \frac{1}{2}\{\mu(x) + \mu(a)\} + 1 - \frac{1}{2}\{\mu(1) + \mu(a)\} = \frac{1}{2}\{\mu(x) + 1\}$ and so $\nu^+(1) = \frac{1}{2}\{\mu(1) + 1\} = 1$. Hence $\nu^+$ is a normal fuzzy implication groupoid of $X$. From $\nu^+(a) > \mu(a)$ it follows that $\mu$ is not maximal. This is contradiction.

**References**

Some Results on $D^*$-fuzzy Cone Metric Spaces and Fixed Point Theorems in Such Spaces

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Abstract:
In this paper, an idea of $D^*$-fuzzy cone metric space is introduced. Some basic definitions viz. convergence of sequence, Cauchy sequence, closedness, completeness etc are given and study some related properties. Some fixed point theorems are established in such spaces.

Keywords:
$D^*$-fuzzy cone metric space, weakly compatible mapping.

0. Introduction

The concept of fuzzy set is introduced by L. A. Zadeh [14] in 1965. After that, to use this concept in topology and analysis different authors have expansively developed the theory of fuzzy sets and its application in different direction.

On the other hand, there have been a number of generalizations of metric spaces. One such generalization is generalized metric space ($D$-metric space) introduced by Dhage [5] in 1992. Many other authors viz. Sedghi et al. [13] made a significant contribution in fixed point theory of $D^*$-metric space (which is similar to $D$-metric space). Recently Sedghi et al. [12] introduced the concept of $M$-fuzzy metric space which is a generalization of fuzzy metric space due to George & Veeramoni [7].

The present author [3] redefined $M$-fuzzy metric space and called it $D^*$-fuzzy metric space and established a decomposition theorem and with the help of this theorem some results were established in such spaces. H. Long-Guang et al [9] generalized the notion of general metric spaces, replacing the real numbers by an ordered Banach space and defined cone metric space. They have proved Banach contraction mapping theorem and some other contraction mapping theorem and some other fixed point theorems of contractive type mappings in cone metric spaces. Subsequently, Rezapour and Hamlbarani [11] contributed some fixed point theorems for contractive type mappings in...
cone metric space. In an earlier paper [4], idea of fuzzy cone metric space has been introduced and some basic definitions are given. Here the range of fuzzy cone metric is considered as ordering fuzzy real numbers defined on a real fuzzy Banach space (Felbin’s type [6]).

The idea of generalized $D'$-metric space is relatively new and it is introduced by Age & Salunke [1]. They have also established some fixed point theorems in such spaces. Following the idea of generalized $D'$-metric space introduced by Age & Salunke [1], in this paper, definition of $D'$-fuzzy cone metric space is given. Some fixed point theorems are established in such space.

The organization of the paper is as follows:

In Section 1, comprises some preliminary results which are used in this paper.

Definition of $D'$-fuzzy cone metric space and some basic properties are discussed in Section 2. In Section 3, some fixed point theorems for contractive mappings are established.

1. Some preliminary results

A fuzzy number is a mapping $x : R \rightarrow [0,1]$ over the set $R$ of all reals.

A fuzzy number $x$ is convex if $x(t) \geq \min \{x(s), x(r)\}$ where $s \leq t \leq r$. If there exists a $t_0 \in R$ such that $x(t_0) = 1$, then $x$ is called normal. For $0 < \alpha \leq 1$, $\alpha$-level set of an upper semi continuous convex normal fuzzy number (denoted by $[\eta]_\alpha$) is a closed interval $[a_\alpha, b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = +\infty$ are admissible. When $a_\alpha = -\infty$, for instance, then $[a_\alpha, b_\alpha]$ means the interval $(-\infty, b_\alpha]$. Similar is the case when $b_\alpha = +\infty$.

A fuzzy number $x$ is called non-negative if $x(t) = 0$, $\forall t < 0$.

Kaleva (Felbin) denoted the set of all convex, normal, upper semicontinuous fuzzy real numbers by $E(R(I))$ and the set of all non-negative, convex, normal, upper semicontinuous fuzzy real numbers by $G(R^*(I))$.

A partial ordering “$\preceq$” in $E$ is defined by $\eta \preceq \delta$ if and only if $a^i_\alpha \leq a^i_\alpha$ and $b^i_\alpha \leq b^i_\alpha$ for all $\alpha \in (0,1]$ where $[\eta]_\alpha = [a^1_\alpha, b^1_\alpha]$ and $[\delta]_\alpha = [a^2_\alpha, b^2_\alpha]$. The strict inequality in $E$ is defined by $\eta < \delta$ if and only if $a^i_\alpha < a^2_\alpha$ and $b^i_\alpha < b^2_\alpha$ for each $\alpha \in (0,1]$.

According to Mizumoto and Tanaka [10], the arithmetic operations $\oplus$, $\ominus$, $\odot$ on $E \times E$ are defined by

\[
\begin{align*}
(x \oplus y)(t) &= \sup_{s \in R} \min \{x(s), y(t-s)\}, \ t \in R \\
(x \ominus y)(t) &= \sup_{s \in R} \min \{x(s), y(s-t)\}, \ t \in R \\
(x \odot y)(t) &= \sup_{s \in R, s \neq 0} \min \left\{ x(s), y\left(\frac{t}{s}\right) \right\}, \ t \in R
\end{align*}
\]
**Proposition 1.1** [10]. Let $\eta, \delta \in E(R(I))$ and $[\eta]_\alpha = [a^\alpha_\alpha, b^\alpha_\alpha]$, $[\delta]_\alpha = [a^\alpha_\alpha, b^\alpha_\alpha]$, $\alpha \in (0,1)$. Then $[\eta \odot \delta]_\alpha = [a^\alpha_\alpha + a^\alpha_\alpha, b^\alpha_\alpha + b^\alpha_\alpha]$, $[\eta \odot \delta]_\alpha = [a^\alpha_\alpha - b^\alpha_\alpha, b^\alpha_\alpha - a^\alpha_\alpha]$, $[\eta \odot \delta]_\alpha = [a^\alpha_\alpha a^\alpha_\alpha, b^\alpha_\alpha b^\alpha_\alpha]$

**Definition 1.1** [8]. A sequence $\{\eta_n\}$ in $E$ is said to be convergent and converges to $\eta$ denoted by $\lim_{n \to \infty} \eta_n = \eta$ if $\lim_{n \to \infty} a^n_\alpha = a_\alpha$ and $\lim_{n \to \infty} b^n_\alpha = b_\alpha$ where $[\eta_n]_\alpha = [a^n_\alpha, b^n_\alpha]$ and $[\eta]_\alpha = [a_\alpha, b_\alpha]$ for all $\alpha \in (0,1)$.

**Note 1.1** [8]. If $\eta, \delta \in G(R(I))$ then $\eta \odot \delta \in G(R(I))$.

**Note 1.2** [8]. For any scalar $t$, the fuzzy real number $t\eta$ is defined as $[t\eta]_\alpha = \begin{cases} t\eta \in (0,1) & \text{if } 0 < t < 1 \\ t\eta \in [0,1) & \text{otherwise} \end{cases}$

**Definition 1.2** [6]. Let $X$ be a vector space over $R$. Let $L, U : [0,1] \times [0,1] \to [0,1]$ be symmetric, nondecreasing in both arguments and satisfy $L(0,0) = 0$ and $U(1,1) = 1$.

Write $[\|x\|]_\alpha = [\|x\|_L, \|x\|_U]$ for $x \in X$, $0 < \alpha \leq 1$ and suppose for all $x \in X$, $x \neq 0$, there exists $\alpha_0 \in (0,1)$ independent of $x$ such that for all $\alpha \leq \alpha_0$,

(A) $\|x\|_\alpha < \infty$

(B) $\inf \|x\|_\alpha > 0$.

The quadruple $(X, \|\|, L, U)$ is called a fuzzy normed linear space and $\|\|$ is a fuzzy norm if

(i) $\|x\|_\alpha = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\|x\|_\alpha$, $x \in X$, $r \in R$;

(iii) for all $x, y \in X$,

(a) whenever $s \leq \|x\|_\alpha$, $t \leq \|y\|_\alpha$ and $s + t \leq \|x + y\|_\alpha$, $\|x + y\|_\alpha (s + t) \geq L(\|x\|_\alpha(s), \|y\|_\alpha(t))$,

(b) whenever $s \geq \|x\|_\alpha$, $t \geq \|y\|_\alpha$ and $s + t \geq \|x + y\|_\alpha$, $\|x + y\|_\alpha (s + t) \leq U(\|x\|_\alpha(s), \|y\|_\alpha(t))$.

**Remark 1.1** [6]. Felbin proved that, if $L = \land (\text{Min})$ and $U = \lor (\text{Max})$ then the triangle inequality (iii) in the Definition 1.1 is equivalent to $\|x + y\|_\alpha \leq \|x\|_\alpha \land \|y\|_\alpha$.

Further $\|\|_i$ for $i = 1, 2$ are crisp norms on $X$ for each $\alpha \in (0,1)$. When $L = \text{min}$ and $U = \text{max}$, then we write $(X, \|\|)$ instead of $(X, L, U, \|\|)$.

**Definition 1.3** [4]. Let $(E, \|\|)$ be a fuzzy real Banach space where $\|\| : E \to R^+(I)$. Denote the range of $\|\|$ by $E^+ (I)$. Thus $E^+ (I) \subset R^+(I)$. 

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Definition 1.4 [4]. A member \( \eta \in R^\ast(I) \) is said to be an interior point if \( \exists r > 0 \) such that 
\[
S(\eta, r) = \{ \delta \in R^\ast(I) : \eta \otimes \delta < r \} \subset E^\ast(I).
\]
Set of all interior points of \( R^\ast(I) \) is called interior of \( R^\ast(I) \).

Definition 1.5 [4]. A subset of \( F \) of \( E^\ast(I) \) is said to be fuzzy closed if for any sequence \( \{ \eta_n \} \) such that \( \lim_{n \to \infty} \eta_n = \eta \) implies \( \eta \in F \).

Definition 1.6 [4]. A subset \( P \) of \( E^\ast(I) \) is called a fuzzy cone if

(i) \( P \) is fuzzy closed, nonempty and \( P \neq \{ 0 \} \);
(ii) \( a, b \in R, a, b \geq 0, \eta, \delta \in P \Rightarrow a\eta \otimes b\delta \in P \);
(iii) \( \eta \in P \) and \( -\eta \in P \Rightarrow \eta = 0 \).

Given a fuzzy cone \( P \subset E^\ast(I) \), define a partial ordering \( \leq \) with respect to \( P \) by \( \eta \leq \delta \) iff \( \delta \ominus \eta \in P \) and \( \eta < \delta \) indicates that \( \eta \leq \delta \) but \( \eta \neq \delta \) while \( \eta \ll \delta \) will stand for \( \delta \ominus \eta \in \text{Int} P \) where \( \text{Int} P \) denotes the interior of \( P \).

The fuzzy cone \( P \) is called normal if there is a number \( K > 0 \) such that for all \( \eta, \delta \in E^\ast(I) \), with \( 0 \leq \eta \leq \delta \) implies \( \eta \leq K\delta \). The least positive number satisfying above is called the normal constant of \( P \).

The fuzzy cone \( P \) is called regular if every increasing sequence which is bounded from above is convergent. That is if \( \{ \eta_n \} \) is a sequence such that \( \eta_1 \leq \eta_2 \leq \cdots \leq \eta_n \leq \cdots \leq \eta \) for some \( \eta \in E^\ast(I) \), then there is \( \delta \in E^\ast(I) \) such that \( \eta_n \to \delta \) as \( n \to \infty \).

Equivalently, the fuzzy cone \( P \) is regular if every decreasing sequence which is bounded below is convergent. It is clear that a regular fuzzy cone is a normal fuzzy cone.

In the following we always assume that \( E \) is a Felbin’s type fuzzy real Banach space, \( P \) is a fuzzy cone in \( E \) with \( \text{Int} P \neq \emptyset \) and \( \leq \) is a partial ordering with respect to \( P \).

Definition 1.7 [4]. Let \( X \) be a nonempty set. Suppose the mapping \( d : X \times X \to E^\ast(I) \) satisfies:

(Fd1) \( \delta \ominus d(x, y) \forall x, y \in X \) and \( d(x, y) = \delta \) iff \( x = y \);
(Fd2) \( d(x, y) = d(y, x) \forall x, y \in X \);
(Fd3) \( d(x, y) \leq d(x, z) \ominus d(z, y) \forall x, y, z \in X \).

Then \( d \) is called a fuzzy cone metric and \( (X, d) \) is called a fuzzy cone metric space.

Definition 1.8. A point \( u \) of a set \( X \) is said to be coincidence point of a pair of mappings \( (f, g) : X \to X \) if there exists \( x \in X \) such that \( f(x) = g(x) = u \).

Definition 1.9. Let \( f \) and \( g \) be two self mappings defined on a set \( X \). \( f \) and \( g \) are called weakly compatible if they commute at coincidence points.
Proposition 1.2 [2]. (M. Abbas et al.) Let \( f \) and \( g \) be weakly compatible self maps of a set \( X \). If \( f \) and \( g \) have a unique point of coincidence \( w = fx = gx \), then \( w \) is the unique common fixed point of \( f \) and \( g \).

2. \( D^* \)-fuzzy cone metric space

In this section an ideal of \( D^* \)-fuzzy cone metric space is introduced and study some properties.

**Definition 2.1.** Let \( X \) be a nonempty set. Suppose that mapping \( D^*: X \times X \times X \rightarrow E^*(I) \) satisfying the following conditions:

1. \( (FD^*1) \) \( D^*(x, y, z) \geq 0 \) \( \forall x, y \in X \);
2. \( (FD^*2) \) \( D^*(x, y, z) = 0 \) iff \( x = y = z \);
3. \( (FD^*3) \) \( D^*(x, y, z) = D^*(p \{x, y, z\}) \) where \( p \) stands for permutation;
4. \( (FD^*4) \) \( D^*(x, y, z) \leq D^*(x, y, a) \oplus D^*(a, z, z) \) \( \forall x, y, z, a \in X \).

Then the function \( D^* \) is called a \( D^* \)-fuzzy cone metric and the pair \( (X, D^*) \) is called a \( D^* \)-fuzzy cone metric space.

**Example 2.1.** Consider the Banach space \( \left( E, \| \cdot \| \right) \) where \( E = \mathbb{R}^2 \) and \( \|x\| = \left( \sum_{i=1}^{2}|x_i|^2 \right)^{1/2} \), \( x = (x_1, x_2) \). Define \( \| \cdot \|: X \rightarrow R^*(I) \) by \( \|x\|(t) = \begin{cases} 1 & \text{if } t > \|x\| \\ 0 & \text{if } t \leq \|x\| \end{cases} \) Then

\( \|x\|_a = \left[ \|x\|, \|y\| \right] \forall a \in (0,1) \). It is easy to verify that,

(i) \( \|x\| = 0 \) iff \( x = 0 \)
(ii) \( \|x + y\| \leq \|x\| \oplus \|y\| \). Thus \( (E, \| \cdot \|) \) is a Felbin’s type \( (L = \min \text{ and } U = \max \) fuzzy normed linear space. Let \( \{(x_n, y_n)\} \) be a Cauchy sequence in \( (X, \| \cdot \|) \). So, \( \lim_{n,m \rightarrow \infty} \|(x_n, y_n) - (x_m, y_m)\| = 0 \Rightarrow \lim_{n,m \rightarrow \infty} \|(x_n - x_m, y_n - y_m)\| = 0 \Rightarrow \{(x_n, y_n)\} \) be a Cauchy sequence in \( (X, \| \cdot \|) \). Since \( (X, \| \cdot \|) \) is complete, \( \exists (x, y) \in X \) such that \( \lim_{n \rightarrow \infty} \|(x_n, y_n) - (x, y)\| = 0 \) i.e. \( \lim_{n \rightarrow \infty} \|(x_n, y_n) - (x, y)\| = 0 \). Thus \( (E, \| \cdot \|) \) is a real fuzzy Banach space. Define \( P = \{\eta \in E^*(I) : \eta \geq 0\} \) (where \( E^*(I) \) is the range of \( \| \cdot \| \)).

(i) \( P \) is fuzzy closed. For, consider a sequence \( \{\delta_n\} \in P \) such that \( \lim_{n \rightarrow \infty} \delta_n \rightarrow \delta \), i.e. \( \lim_{n \rightarrow \infty} \delta_{n,\alpha} = \delta_{\alpha} \) and \( \lim_{n \rightarrow \infty} \delta_{2,\alpha} = \delta_{2,\alpha} \) where \( \left[ \delta_n \right]_{\alpha} = \left[ \delta_{1,\alpha}, \delta_{2,\alpha} \right] \) and \( \left[ \delta \right]_{\alpha} = \left[ \delta_{1,\alpha}, \delta_{2,\alpha} \right] \) \( \forall \alpha \in (0,1] \). Now \( \delta_n \geq 0 \forall n \). So, \( \delta_{1,n,\alpha} \geq 0 \text{ and } \delta_{2,n,\alpha} \geq 0 \) \( \forall \alpha \in (0,1] \). \( \Rightarrow \lim_{n \rightarrow \infty} \left[ \delta_n \right]_{\alpha} = \left[ \delta_{\alpha}, \delta_{\alpha} \right] \) \( \forall \alpha \in (0,1] \).
\[ \delta_{n,\alpha} \geq 0 \] and \( \lim_{n\to\infty} \delta_{n,\alpha} \geq 0 \) \( \forall \alpha \in (0,1] \) \( \Rightarrow \delta_{n,\alpha} \geq 0 \) and \( \delta_{n,\alpha} \geq 0 \) \( \forall \alpha \in (0,1] \) \( \Rightarrow \delta \geq 0 \).

So \( \delta \in P \). Hence \( P \) is fuzzy closed.

(ii) It is obvious that, \( a, b \in R \), \( a, b \geq 0 \) \( \eta, \delta \in P \Rightarrow a \eta \oplus b \delta \in P \).

(iii) Let \( \eta \in P \). If \( \eta \in P \), then for all \( t < 0 \) we have \( (-\eta)(t) = \eta(-t) = \eta(s) \geq 0 \) for \( s = -t > 0 \). If \( \eta(s) = 0 \) \( \forall s > 0 \) then \( \eta = \bar{0} \). Otherwise for some \( s = -t > 0 \), \( \eta(s) > 0 \) i.e. for some \( t < 0 \), \( (-\eta)(t) > 0 \). In that case \( -\eta \) does not belong to \( P \).

Hence \( \eta \in P \) and \( -\eta \in P \) implies \( \eta = 0 \). Thus \( P \) is a fuzzy cone in \( E \). Take \( X = R \) and choose the ordering \( \leq \) as \( \leq \). Define \( D^* : X \times X \times X \to E^* (I) \) by \( D^*(x,y,z)(t) = \frac{1}{t} \begin{cases} (|x-y| + |y-z| + |z-x|) & \text{if } t \geq 1 \\ 0 & \text{if } t < 1 \end{cases} \) \( \alpha \)-level sets are given by \( \big[ D^*(x,y,z) \big]_{\alpha} = \left[ |x-y| + |y-z| + |z-x|, |x-y| + |y-z| + |z-x|/\alpha \right] \). Now we show that \( D^* \) satisfies the conditions \( (FD^1)-(FD^4) \). We have, \( D_a^1(x,y,z) = |x-y| + |y-z| + |z-x| \) and \( D_a^2(x,y,z) = (|x-y| + |y-z| + |z-x|)/\alpha \) \( \forall \alpha \in (0,1] \). Conditions \( (FD^1), (FD^2) \) and \( (FD^3) \) can be easily verified.

Hence \( D^* \) is a \( D^* \)-fuzzy cone metric and \( (X,D^*) \) is a \( D^* \)-fuzzy cone metric space.

**Definition 2.1.** Let \( (X,D^*) \) be a \( D^* \)-fuzzy cone metric space. Let \( \{x_n\} \) be a sequence in \( X \) and \( x \in X \). If for every \( c \in E \) with \( \bar{0} \ll \|c\| \) there is a positive integer \( N \) such that for all \( n \geq N \), \( D^*(x_n, x) \ll \|c\| \), then \( \{x_n\} \) is said to be convergent and converges to \( x \) and \( x \) is called the limit of \( \{x_n\} \). We denote it by \( \lim_{n \to \infty} x_n = x \).

**Proposition 2.1.** Let \( (X,D^*) \) be a \( D^* \)-fuzzy cone metric space. Then for all \( x, y, z \in X \), \( D^*(x,y,z) = D^*(x,y) \).

**Proof.** We have

\[ D^*(x,y,z) \leq D^*(x,x) \oplus D^*(y,y) = D^*(x,y,y) \]  

(i)

Again \( D^*(y,y,x) \leq D^*(y,y,y) \oplus D^*(y,x,x) = D^*(y,x,x) \) i.e.

\[ D^*(x,y,y) \leq D^*(x,y,y) \]  

(ii)

From (i) and (ii) we get, \( D^*(x,y,y) = D^*(x,x,y) \).
Lemma 2.1. Let \((X,D^*)\) be a \(D^*\)-fuzzy cone metric space and \(P\) be a normal fuzzy cone with normal constant \(K\). Let \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) converges to \(x\) iff \(D^*(x_m,x_n,x) \to \bar{0}\) as \(n \to \infty\).

**Proof.** First we suppose that \(\{x_n\}\) converges to \(x\). For every real \(\varepsilon > 0\), choose \(c \in E\) with \(\bar{0} \ll \|c\|\) and \(K\|c\| < \varepsilon\). Then \(\exists\) a natural number \(N\), such that \(\forall m, n \geq N\), \(D^*(x_m,x_n,x) \ll \|c\|\). So that when \(m, n \geq N\), \(D^*(x_m,x_n,x) \leq K\|c\| < \varepsilon\) (since \(P\) is normal). \(\Rightarrow D^*_{\alpha}(x_m,x_n,x) < \varepsilon\), \(D^*_\alpha(x_m,x_n,x) < \varepsilon\) \(\forall m, n \geq N\), \(\forall \alpha \in (0,1]\) \(\Rightarrow \lim_{m,n \to \infty} D^*_{\alpha}(x_m,x_n,x) = 0\) and \(\lim_{m,n \to \infty} D^*_\alpha(x_m,x_n,x) = 0\) \(\forall \alpha \in (0,1]\) \(\Rightarrow \lim_{m,n \to \infty} D^*(x_m,x_n,x) = \bar{0}\).

Conversely, suppose that \(\lim_{m,n \to \infty} D^*(x_m,x_n,x) = \bar{0}\). Let \(c \in E\) with \(\bar{0} \ll \|c\|\). Choose \(\delta > 0\) such that \(\|c\| < 2\delta\). This implies that \(\|c\| = \|c\| < 2\delta\) i.e. \(\|c\| \ll \|c\|\in \text{Int} P\). For this \(\delta\), there is a positive integer \(N\) such that \(\forall m, n \geq N\), \(D^*(x_m,x_n,x) < 2\delta\). Let \(D^*(x_m,x_n,x) = \|y_{m,n}\|\) where \(y_{m,n} \in E\). So \(\|y_{m,n}\| < 2\delta\) \(\forall m, n > N\). So, \(\|c\| = \|y_{m,n}\| \in \text{Int} P\) \(\forall m, n \geq N\) \(\Rightarrow \|y_{m,n}\| \ll \|c\|\) \(\forall m, n \geq N\) \(\Rightarrow D^*(x_m,x_n,x) < \|c\|\) \(\forall m, n \geq N\) \(\Rightarrow D^*(x_m,x_n,x) \to \bar{0}\) as \(m, n \to \infty\) (since \(c \in E\) is arbitrary).

**Lemma 2.2.** If \((X,D^*)\) is a \(D^*\)-fuzzy cone metric space, then the following results are equivalent.

(i) \(\{x_n\}\) converges to \(x\).

(ii) \(D^*(x_n,x,x) \to \bar{0}\) as \(n \to \infty\).

(iii) \(D^*(x_n,x,x) \to \bar{0}\) as \(n \to \infty\).

**Proof.** Suppose (i) holds. From Lemma 3.1, we have \(D^*(x_n,x_n,x) = \bar{0}\) as \(m, n \to \infty\).

It follows that, \(D^*(x_n,x_n,x) \to \bar{0}\) as \(n \to \infty\). So (ii) holds.

Assume (ii) holds. i.e. \(D^*(x_n,x,x) \to \bar{0}\) as \(n \to \infty\). We have

\[D^*(x_n,x,x) = \bar{0}\] (i)

Again \(D^*(y,y,x) = \bar{0}\). i.e.

\[D^*(y,y,x) = \bar{0}\] (ii)

From (i) and (ii), we get \(D^*(x_n,x,y) = \bar{0}\). So \(D^*(x_n,x_n,x) = D^*(x_n,x,x)\). Thus \(D^*(x_n,x_n,x) \to \bar{0}\) as \(n \to \infty\) \(\Rightarrow D^*(x_n,x_n,x) \to \bar{0}\) as \(n \to \infty\). Hence (ii) \(\Rightarrow\) (iii).

Next suppose (iii) holds and we have to show that (iii) \(\Rightarrow\) (i). Since \(D^*(x_n,x,x) \to \bar{0}\) as \(n \to \infty\) and \(D^*(x_n,x_n,x) = D^*(x_n,x,x)\). We have \(D^*(x_n,x_n,x) \to \bar{0}\) as \(n \to \infty\).

Then for any \(c \in E\) with \(\bar{0} \ll \|c\|\) there is a positive integer \(N\) such that \(\forall n \in N\),
\[D'(x_n,x) \leq \|e/2\| \quad \text{and} \quad D'(x_n,x) = D'(x_n,x) \leq D'(x_n,x) \oplus D'(x_n,x) \quad \text{i.e.} \quad D'(x_n,x) \leq \|e/2\|, \quad \|e/2\| = \|e\|. \quad \text{i.e.} \quad \{x_n\} \text{ is convergent. This completes the proof.}

**Lemma 2.3.** Let \((X,D')\) be a \(D'\)-fuzzy cone metric space and \(P\) be a normal cone with normal constant \(K\). Let \(\{x_n\}\) be a sequence in \(X\). If \(\{x_n\}\) is convergent then its limit is unique.

**Proof.** If possible suppose that \(\lim_{n \to \infty} x_n = x\) and \(\lim_{n \to \infty} x_n = y\). For a given \(\varepsilon > 0\), choose \(c \in E\) with \(0 \ll \|e\|\) and \(K \|e\| < \varepsilon\). Then there exists a natural number \(N\) such that
\[
\forall n \geq N, \quad D'(x_n,x) \leq \|e/2\| \quad \text{and} \quad D'(x_n,x) \leq \|e/2\|. \quad \text{We have} \quad D'(x_n,x) \leq D'(x_n,x) \oplus D(x_n,y) = D'(x_n,x) \oplus d(x_n,y). \quad \text{Since} \quad X\text{ is a normal cone with normal constant} \quad K, \quad \text{we have} \quad D'(x_n,x) \leq K \|e\|. \quad \text{This implies that} \quad D'(x_n,x) \leq \varepsilon. \quad \text{i.e.} \quad D'_1(x_n,x) < \varepsilon \quad \text{and} \quad D'_2(x_n,x) < \varepsilon. \quad \forall \alpha \in (0,1]. \quad \text{Since} \quad \varepsilon > 0 \quad \text{is arbitrary, we have} \quad D'_1(x_n,x) = 0 \quad \text{and} \quad D'_2(x_n,x) = 0. \quad \forall \alpha \in (0,1]. \quad \text{Hence} \quad D(x_n,x) = 0. \quad \text{Thus} \quad x = y.

**Definition 2.2.** Let \((X,D')\) be a \(D'\)-fuzzy cone metric space. A sequence \(\{x_n\}\) in \(X\) is said to be a Cauchy sequence if for any \(c \in E\) with \(0 \ll \|e\|\), there exists an \(\varepsilon > 0\) such that \(\forall n,m \geq N, \quad D'(x_n,x_m) \ll \|e\|. \quad \text{This means} \quad \{x_n\} \text{ is a Cauchy sequence.}

**Definition 2.4.** Let \((X,D')\) be a \(D'\)-fuzzy cone metric space. If every Cauchy sequence in \(X\) is convergent in \(X\), then \(X\) is called a complete \(D'\)-fuzzy cone metric space.

**Lemma 2.4.** Let \((X,D')\) be a \(D'\)-fuzzy cone metric space and \(\{x_n\}\) be a sequence in \(X\). If \(\{x_n\}\) is convergent then it is a Cauchy sequence.

**Proof.** Let \(\{x_n\}\) converges to \(x\). So for any \(c \in E\) with \(0 \ll \|e\|\) there exists a natural number \(N\) such that \(\forall l,m,n \geq N, \quad D'(x_n,x) \ll \|e/2\| \quad \text{and} \quad D'(x_n,x) \ll \|e/2\|. \quad \text{Thus} \quad D'(x_n,x) \ll \|e/2\| \quad \forall l,m,n \geq N. \quad \text{Thus} \quad \{x_n\} \text{ is a Cauchy sequence.}

**Lemma 2.5.** Let \((X,D')\) be a \(D'\)-fuzzy cone metric space, \(P\) be a normal fuzzy cone with normal constant \(K\). Let \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) is a Cauchy sequence if \(D'(x_n,x) \to 0\) as \(l,m,n \to \infty\).

**Proof.** Let \(\{x_n\}\) be a Cauchy sequence in \(X\). For \(\varepsilon > 0\) choose \(c \in E\) with \(0 \ll \|e\|\) such that \(K \ll \varepsilon\). Then there exists a natural number \(N\) such that \(\forall l,m,n \geq N, \quad D'(x_n,x) \ll \|e/2\| \quad \text{i.e.} \quad \forall l,m,n \geq N, \quad D'(x_n,x) \ll \|e/2\|. \quad \text{So that} \quad \forall l,m,n \geq N, \quad D'(x_n,x) \ll \|e/2\| \quad \text{since} \quad P \quad \text{is normal}. \quad \text{Since} \quad \varepsilon > 0 \quad \text{is arbitrary, it follows that} \quad D'(x_n,x) \ll \|e/2\|. \quad \text{Since} \quad \varepsilon > 0 \quad \text{is arbitrary,
for all $x_n, x_i \to \overline{0}$ as $l, m, n \to \infty$. Conversely suppose that $D'(x_n, x_i) \to \overline{0}$ as $l, m, n \to \infty$. For $c \in E$ with $\overline{0} \preceq \|c\|$, there is $\delta > 0$ such that \[\|x\| \preceq \delta \] i.e. $\|c\| \odot (\|c\| \odot \|x\|) \preceq \delta$ implies $\|c\| \odot \|x\| \in \text{Int}P$. For this $\delta > 0$, there exists a natural number $N$ such that $\forall i, m, n \geq N$, $D'(x_n, x_i) \preceq \delta$. i.e. $\|x_m, n, i\| \preceq \delta$ if we write $D'(x_m, n, i) = \|x_m, n, i\|$.

3. Fixed point theorems in $D'$-fuzzy cone metric spaces

In this Section some fixed point theorems of self mappings are established in $D'$-fuzzy cone metric spaces.

Theorem 3.1. Let $(X, D')$ be a $D'$-fuzzy cone metric space, $P$ be a normal fuzzy cone with normal constant $K$. Let $S, T : X \to X$ be two self mappings satisfying the following conditions:

(i) $T(X) \subset S(X)$

(ii) $T(X)$ or $S(X)$ is complete and
(iii) \(D'(Tx, Ty, Tz) \leq aD'(Sx, Sy, Sz) + bD'(Sr, Tr, Tz) + cD'(Sy, Ty, Ty) + dD'(Sz, Tz, Tz)\) \(\forall x, y, z \in X\) where \(a, b, c, d \geq 0\), \(a + b + c + d < 1\). Then \(S\) and \(T\) have a unique point of coincidence in \(X\). Moreover if \(S\) and \(T\) are weakly compatible, then \(S\) and \(T\) have a unique common fixed point.

**Proof.** Let \(x_0\) be arbitrary. There exists \(x_1 \in X\) such that \(Tx_0 = Sx_1\). In this way we get a sequence \(\{Sx_n\}\) with \(Tx_n = Sx_{n+1}\). We have, \(D'(Sx_n, Sx_{n+1}, Sx_{n+1}) = D'(Tx_{n-1}, Tx_n, Tx_n)\). Thus by (iii) we have, \(D'(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq aD'(Sx_{n-1}, Sx_n, Sx_n) + bD'(Sx_{n-1}, Tx_n, Tx_n) + cD'(Sx_n, Tx_{n-1}, Tx_n) + dD'(Sx_{n-1}, Sx_n, Sx_{n+1})\) i.e. \(D'(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq aD'(Sx_{n-1}, Sx_n, Sx_n) + bD'(Sx_{n-1}, Sx_n, Sx_n) + cD'(Sx_{n-1}, Sx_n, Sx_n) + dD'(Sx_{n-1}, Sx_n, Sx_{n+1})\). This implies that, \(D'(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq qD'(Sx_{n-1}, Sx_n, Sx_n)\) where \(q = a + b/1 - (c + d)\) and also \(0 < q < 1\). By repeated application of above inequality we have,

\[D'(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq q^n D'(Sx_0, Sx_1, Sx_1)\]  

(1)

Then for all \(m, n \in N, m > n\) and by using (FN4) we have, \(D'(Sx_n, Sx_m, Sx_m) = D'(Sx_n, Sx_m, Sx_m) \leq D'(Sx_n, Sx_n, Sx_n) + D'(Sx_m, Sx_m, Sx_m)\) i.e. \(D'(Sx_n, Sx_m, Sx_m) \leq D'(Sx_n, Sx_n, Sx_n)\). i.e.

\[D'(Sx_n, Sx_m, Sx_m) \leq D'(Sx_n, Sx_n, Sx_n) + D'(Sx_n, Sx_m, Sx_m)\]  

(2)

In similar way we get,

\[D'(Sx_{n+1}, Sx_m, Sx_m) \leq D'(Sx_{n+1}, Sx_{n+1}, Sx_{n+2}) + D'(Sx_{n+2}, Sx_m, Sx_m)\]  

(3)

\[\cdots\]

\[D'(Sx_{m-1}, Sx_m, Sx_m) \leq D'(Sx_{m-1}, Sx_{m-1}, Sx_m) + D'(Sx_{m-1}, Sx_m, Sx_m)\]  

(4)

From (2), (3) and (4) we have, \(D'(Sx_n, Sx_m, Sx_m) \leq D'(Sx_{n+1}, Sx_{n+1}, Sx_{n+1}) + D'(Sx_{n+2}, Sx_{n+2}, Sx_{n+2}) + \cdots + D'(Sx_{m-1}, Sx_{m-1}, Sx_{m-1}) + D'(Sx_{m-1}, Sx_m, Sx_m)\) i.e. \(D'(Sx_n, Sx_m, Sx_m) \leq D'(Sx_{n+1}, Sx_{n+1}, Sx_{n+1}) + D'(Sx_{n+2}, Sx_{n+2}, Sx_{n+2}) + \cdots + D'(Sx_{m-1}, Sx_{m-1}, Sx_{m-1}) + D'(Sx_{m-1}, Sx_m, Sx_m)\) i.e. \(D'(Sx_n, Sx_m, Sx_m) \leq (q^n + q^{n+1} + \cdots + q^{m-1}) D'(Sx_{n+1}, Sx_{n+1}, Sx_{n+1})\) i.e. \(D'(Sx_n, Sx_m, Sx_m) = q^n (1 - q) D'(Sx_{n+1}, Sx_{n+1}, Sx_{n+1})\). Since \(P\) is a normal cone with normal constant \(K\) we have, \(D'(Sx_n, Sx_m, Sx_m) \leq K \frac{q^n}{1-q} D'(Sx_{n+1}, Sx_{n+1}, Sx_{n+1})\).
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$\Rightarrow D^i_\alpha(Sx_n, Sx_m, Sx_m) \leq \frac{q^n}{1-q} D^i_\alpha(Sx_0, Sx_0, Sx_1)$ and $D^2_\alpha(Sx_n, Sx_m, Sx_m) \leq \frac{q^n}{1-q} D^2_\alpha(Sx_0, Sx_0, Sx_1)$ \quad $\forall \alpha \in (0,1) \Rightarrow \lim_{m,n \to \infty} D^i_\alpha(Sx_n, Sx_m, Sx_m) = 0$ and $\lim_{m,n \to \infty} D^2_\alpha(Sx_n, Sx_m, Sx_m) = 0$ \quad $\forall \alpha \in (0,1)$ (since $q < 1$) (5) Again for $l, m, n \in N$, we have $D'(Sx_n, Sx_m, Sx_m) \leq D'(Sx_n, Sx_m, Sx_m) \oplus D'(Sx_m, Sx_m, Sx_m)$. Since $P$ is a normal cone we have, $D'(Sx_n, Sx_m, Sx_m) \leq K \{ D'(Sx_n, Sx_m, Sx_m) \oplus D'(Sx_m, Sx_m, Sx_m) \}$ \quad $\Rightarrow D^i_\alpha(Sx_n, Sx_m, Sx_m) \leq K D^i_\alpha(Sx_n, Sx_m, Sx_m) \oplus K D^i_\alpha(Sx_m, Sx_m, Sx_m)$ and $D^2_\alpha(Sx_n, Sx_m, Sx_m) \leq K D^2_\alpha(Sx_n, Sx_m, Sx_m) \oplus K D^2_\alpha(Sx_m, Sx_m, Sx_m)$ \quad $\forall \alpha \in (0,1) \Rightarrow \lim_{m,n \to \infty} D^i_\alpha(Sx_n, Sx_m, Sx_m) = 0$ and $\lim_{m,n \to \infty} D^2_\alpha(Sx_n, Sx_m, Sx_m) = 0$ \quad $\forall \alpha \in (0,1)$ by (5) \quad $\Rightarrow \lim_{m,n \to \infty} D'(Sx_n, Sx_m, Sx_m) = 0$. Thus $\{Sx_n\}$ is a Cauchy sequence in $(X, D')$. Since $S(X)$ is complete, $\exists u \in S(X)$ such that $\{Sx_n\} \to u$ as $n \to \infty$ and there exists $p \in X$ such that $Sp = u$. If $T(X)$ is complete, then there exists $u \in T(X)$ such that $Sx_n \to u$. As $T(X) \subset S(X)$, we have $u \in S(X)$. Then there exists $p \in X$ such that $Sp = u$. We claim that $Tp = u$. Now, $D'(Tp, Tp, u) \leq D'(Tp, Tp, Tp) \oplus D'(Tp, Tp, Tp) \oplus D'(Tp, Tp, Tp)$ \quad $\forall \alpha \in (0,1)$ \quad $\Rightarrow \lim_{m,n \to \infty} D'(Tp, Tp, Tp) = 0$ and $\lim_{m,n \to \infty} D'(Tp, Tp, Tp) = 0$ \quad $\forall \alpha \in (0,1)$ by (5). Hence $D'(Tp, Tp, u) = 0$. Thus $p$ is a point of coincidence point of $S$ and $T$. Now we show that $S$ and $T$ have a unique point of coincidence. Assume that there exists a point $q$ on $X$ such that $Sq = Tq$. Now, $D'(Tp, Tp, Tq) \leq aD'(Sp, Sp, Sp) \oplus bD'(Sp, Sp, Tp) \oplus cD'(Sp, Sp, Tp)$ \quad $\forall \alpha \in (0,1)$ \quad $\Rightarrow \lim_{m,n \to \infty} D'(Tp, Tp, Tq) = 0$ and $\lim_{m,n \to \infty} D'(Tp, Tp, Tq) = 0$ \quad $\forall \alpha \in (0,1)$ by (5). Since $a \leq 0$, it implies that $D'(Tp, Tp, Tq) = 0$, i.e. $Tp = Tq$. Thus $p$ is a unique point of coincidence of $S$ and $T$. By Proposition 1.2, it follows that $S$ and $T$ have a unique common fixed point.

**Theorem 3.2.** Let $(X, D')$ be a complete $D'$-fuzzy cone metric space, $P$ be a normal fuzzy cone with normal constant $K$ and let $T : X \to X$ be a mapping satisfies the following conditions:
Then $T$ has a unique fixed point in $X$.

Proof. Proof follows from the Theorem 4.1 by replacing $S$ by identity mapping.

Conclusion

In this paper, an idea of $D^*$-fuzzy cone metric space is introduced which is a generalization of $D^*$-fuzzy metric space. In fuzzy cone metric space, range of fuzzy metric is considered as ordering fuzzy real numbers defined on a real fuzzy Banach space. It is seen that Kaleva et al. type (max, min) fuzzy metric space is a particular case of fuzzy cone metric space. I think that there is a large scope of developing more results of fuzzy functional analysis in this context.

References

New Distance and Similarity Measures for Soft Sets

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Abstract:
In this paper, we will show that some results presented in [3] may be unreasonable and proper results will be introduced. Meanwhile new distance measure and similarity measures for soft sets will be proposed. Furthermore, some application examples about soft sets are also given.

Keywords:
Soft set, Distance measure, Similarity measure

1. Introduction

In order to handle the uncertain problem that exist in engineering, management, economic and medical sciences, researchers have proposed probability theory, fuzzy set theory [5], rough set theory [6], intuitionistic fuzzy set theory [7] etc. They are convenient to describe uncertainties and help us to handle lots of practical problems in our life. However, these approaches have their limitations [1]. In 1999, Molodtsov [1] introduced the soft set theory, it is a new mathematical tool for dealing with uncertainties which traditional mathematical tools cannot handle. And it is free from the inadequacy arising form the theories mentioned above.

Currently, research work on soft sets has made great progress. Maji et al. [9] introduced the application of soft sets to a decision making problem. Chen et al. [13] proposed a new definition of parameterization reduction of soft sets and compared it with the concept of attributes reduction in rough sets theory. Maji et al. [2] proposed some operations on soft sets and made a theoretical study of the soft set theory in more detail. M. Irfan et al. [12] also gave some new operations such as the restricted intersection, the restricted union, and the extended intersection of two soft sets. At the same time, they improved the notion of complement of soft set. Aktas and Cagman [15] compared soft set to the related concepts of fuzzy sets and rough sets. They also gave a definition of soft groups and derived their basic properties. Jun [16] introduced the notion of soft $BCK/BCI$ -algebras and soft subalgebras, and derived their basic properties. Babitha et al. [14] discussed the transitive closures, and orderings on a soft set is defined and also
obtain some set theoretical results. Qin et al. [10] presented the concept of soft equality and introduced some related properties. Feng et al. [11] introduced the notion of soft rough sets and soft rough fuzzy set.

Majumdar and Samanta initiated the study of uncertainty measures for soft sets. They proposed some distance measures for soft sets with the same parameter sets in [4], and also introduced similarity measure for soft sets. Kharal introduced a general approach to study the distance measure and similarity measures of soft sets in [3]. Yang has also proposed new similarity measure in [8].

In this paper, we will present some counterexamples to show that some propositions in [3] may be unreasonable and proper results will be introduced. Furthermore, in Section 4, new distance measures and similarity measures are proposed for soft sets. Meanwhile some propositions and example of its application will be introduced.

2. Preliminaries

In this section, we review some basic notions in soft set theory. Let $U$ be an initial universe set and $E$ be the set of all possible parameters under consideration with respect to $U$. The set of all subsets of $U$ is denoted by $P(U)$. Molodtsov [1] defined the notion of a soft set in the following way.

**Definition 1** [1]. A pair $(F,A)$ is called a soft set over $U$, where $A \subseteq E$, and $F$ is a mapping given by $F : A \rightarrow P(U)$.

In other words, the soft set is a parameterized family of subsets of the set $U$. For any parameter $A \in E$, $(F(A))$ may be considered as the set of $A$-approximate elements of the soft set $(F,A)$.

**Definition 2** [3]. Let $U$ be a universe and $E$ be a set of parameters. Then the pair $(U,E)$, called a soft space, is the collection of all soft sets on $U$ with parameters from $E$.

**Definition 3** [2]. For two soft sets $(F,A)$ and $(G,B)$ over $U$, we say that $(F,A)$ is a soft subset of $(G,B)$, if

(i) $A \subseteq B$;

(ii) $\forall A, F(A) \subseteq G(A)$.

We write $(F,A) \subseteq (G,B)$. $(F,A)$ is said to be a soft super set of $(G,B)$, if $(G,B)$ is a soft subset of $(F,A)$.

**Definition 4** [12]. The complement of a soft set $(F,A)$ is denoted by $(F,A)^c$, and is defined by $(F,A)^c = (F^c,A)$, where $F^c : A \rightarrow P(U)$ is a mapping given by $F^c(A) = F(A)^c$, $\forall A \in E$.
Definition 5 [3]. A mapping $S: (U, E) \times (U, E) \rightarrow [0,1]$ is said to be similarity measure if its values $S(F, (G, B))$, for arbitrary soft sets $(F, A)$ and $(G, B)$ in the soft space $(U, E)$, satisfies following axioms:

(S1) $0 \leq S((F, A), (G, B)) \leq 1$;

(S2) $S((F, A), (F, A)) = 1$;

(S3) $S((F, A), (G, B)) = S((G, B), (F, A))$;

(S4) If $(F, A) \preceq (G, B) \preceq (H, C)$, then $S((F, A), (H, C)) \leq S((F, A), (G, B))$ and $S((F, A), (H, C)) \leq S((G, B), (H, C))$.

Definition 6 [3]. Let $(F, A), (G, B)$ and $(H, C)$ be soft sets in a soft space $(U, E)$ and $d: X \times X \rightarrow R^+$ a mapping. Then

(1) $d$ is said to be quasi-metric if it satisfies

$(M_1)$ $d((F, A), (G, B)) \geq 0$.

$(M_2)$ $d((F, A), (G, B)) = d((G, B), (F, A))$.

(2) A quasi-metric $d$ is said to be semi-metric if

$(M_3)$ $d((F, A), (G, B)) + d((G, B), (H, C)) \geq d((F, A), (H, C))$.

(3) A semi-metric $d$ is said to be pseudo metric if

$(M_4)$ $(F, A) = (G, B) \Rightarrow d((F, A), (G, B)) = 0$.

(4) A pseudo metric $d$ is said to be metric if

$(M_5)$ $d((F, A), (G, B)) = 0 \Rightarrow (F, A) = (G, B)$.

Definition 7 [2]. Denote the absolute null and absolute whole soft sets in a soft space $(U, E)$ as $(F_\emptyset, E), (F_U, E)$ respectively, they have been defined as $F_\emptyset(e) = \emptyset$, $F_U(e) = U$ for each $e \in E$.

3. Counterexamples

In this section, we will show that some Lemma and propositions presented in [3] are not reasonable. And we will modify them.

Symmetric difference between two sets $A$ and $B$ is denoted and defined as:

$$A \Delta B = (A \cup B) \setminus (A \cap B).$$

Definition 8 [3]. For two soft sets $(F, A)$ and $(G, B)$ in a soft space $(U, E)$, where $A \cup B \neq \emptyset$, we define Euclidean distance as:
Normalized Euclidean distance as:

\[
e((F, A), (G, B)) = |A \Delta B| + \sqrt{\sum_{e \in A \Delta B} |F(e)\Delta G(e)|^2}
\]

where \(\chi(e) = |F(e)\Delta G(e)|^2 / \sqrt{F(e) \cup G(e)}\) if \(F(e) \cup G(e) \neq \emptyset\); \(\chi(e) = 0\) otherwise.

**Lemma 9** [3]. For the soft sets \((F_0, E), (F_U, E)\) and an arbitrary soft set \((F, A)\) in a soft space \((U, E)\), we have:

1. \(e((F, A), (F, A)') = 2|A|\).
2. \(q((F, A), (F, A)') = \sqrt{2}|A|\).
3. \(e((F_0, E), (F_U, E)) = \sqrt{|E||U|}\).
4. \(q((F_0, E), (F_U, E)) = \sqrt{|E||U|}\).

**Example 1.** Let \((U, E)\) be a soft space with \(U = \{a, b, c, d\}\) and \(E = \{e_1, e_2, e_3, e_4\}\), suppose \((F, A)\) is a soft set defined by: \(A = \{e_1, e_2, e_3\}, F(e_1) = \{a, d\}, F(e_2) = \{b, c\}, F(e_3) = \{a, b, c\}\). Then \(F^c(e_1) = \{b, c\}, F^c(e_2) = \{a, d\}, F^c(e_3) = \{d\}\). Hence

\[
e((F, A), (F, A)') = |A \Delta A| + \sqrt{\sum_{e \in A \Delta A} |F(e)\Delta F^c(e)|^2} = \sqrt{48} \neq 2|A| = 8.
\]

\[
q((F, A), (F, A)') = \frac{|A \Delta A|}{\sqrt{|A \cup A|}} + \sqrt{\sum_{e \in A \Delta A} \chi(e)} = \sqrt{12} \neq 2|A| = 8.
\]

By this Example, (3) and (4) in Lemma 9 are not suitable. In fact, we have the following theorem.

**Theorem 10.** For the soft sets \((F_0, E), (F_U, E)\) and an arbitrary soft set \((F, A)\) in a soft space \((U, E)\), we have:

1. \(e((F, A), (F, A)') = |X|\sqrt{|A|}\).
2. \(q((F, A), (F, A)') = \sqrt{|X||A|}\).
3. \(e((F_0, E), (F_U, E)) = |U|\sqrt{|E|}\).
4. \(q((F_0, E), (F_U, E)) = \sqrt{|U||E|}\).
Proof. (1') For each \( e \in A \), we have \( F(e) \Delta F^c(e) = (F(e) \cup F^c(e)) \setminus (F(e) \cap F^c(e)) = U \). And consequently \( c((F, A), (F, A)^c) = |A \Delta A| + \sqrt{\sum_{e \in A} |F(e) \Delta F^c(e)|^2} = |U| \sqrt{|A|} \).

(4') For any \( e \in E \), by \( F_{\emptyset}(e) \cup F_{\chi}(e) = X \neq \emptyset \), we have \( \chi(e) = \frac{|F_{\emptyset}(e) \Delta F_{\chi}(e)|^2}{|F_{\emptyset}(e) \cup F_{\chi}(e)|} = \frac{|F_{\emptyset}(e) \cup F_{\chi}(e)|}{|F_{\emptyset}(e) \cup F_{\chi}(e)|} = \frac{|U|}{|E|} = |U| \). And consequently

\[
q((F_{\emptyset}, E), (F_{\chi}, E)) = \frac{|E \Delta E|}{\sqrt{|E \cup E|}} + \sqrt{\sum_{e \in A \cup \chi} \left| \frac{|F_{\emptyset}(e) \Delta F_{\chi}(e)|^2}{|F_{\emptyset}(e) \cup F_{\chi}(e)|} \right|} = \sqrt{|E|} \sqrt{||U|}.
\]

(2'), (3') can be proven in a similar way as (1'), (4').

**Definition 11** [3]. For two soft sets \((F, A)\) and \((G, B)\) in a soft space \((U, E)\), we define a set theoretic matching function similarity measure as:

\[
S((F, A), (G, B)) = \frac{|A \cap B|}{\max\{|A|, |B|\}} + \sum_{e \in A \cup B} \frac{|F(e) \cap G(e)|}{\max\{|F(e)|, |G(e)|\}}.
\]

**Proposition 12** [3]. For an arbitrary soft set \((F, A)\) in a soft space \((U, E)\), we have

\[
S\left((F, A), (F, A)^c\right) = 0.
\]

By the Definition 8, we can get the Proposition 12 is not correct, it should be modified as \( S\left((F, A), (F, A)^c\right) = 1 \).

4. New distance measure and new similarity measure

4.1 New distance measure

Majumdar [4] have proposed distance measure for soft sets, but those distance measures for soft sets with the same parameter set \( E \). The measure is a partial measure. Kharal thought that distance measures and similarity measures for soft sets should be related between both the values of parameters and the mapping on it. Hence we define distance measures for soft sets in the following way:

**Definition 13.** Let \((F, A)\) and \((G, B)\) be soft sets in a soft space \((U, E)\), \n\( d : (U, E) \times (U, E) \rightarrow R \) be a mapping \( d \) is called a distance measure if it satisfied the following conditions:

1. \( d\left((F, A), (G, B)\right) \geq 0 \);
(2) \( d((F,A),(G,B)) = d((G,B),(F,A)) \);

(3) \( d((F,A),(G,B)) = 0 \) if and only if \((F,A) = (G,B)\);

(4) If \((F,A) \subseteq (G,B) \subseteq (H,C)\), then \( d((F,A),(G,B)) \leq d((F,A),(H,C)) \) and \( d((G,B),(H,C)) \leq d((F,A),(H,C)) \).

**Theorem 14.** \( d((F,A),(G,B)) \) is a distance measure, where

\[
d((F,A),(G,B)) = |AB| + \sum_{e \in A \cap B} \chi(e)
\]

\[
\chi(e) = \frac{|F(e)\Delta G(e)|}{|F(e) \cup G(e)|} \text{ if } |F(e) \cup G(e)| \neq 0 \text{ and } \chi(e) = 0 \text{ otherwise.}
\]

**Proof.** (1), (2) and (3) are trivial.

(4) When \(|F(e) \cup G(e)| \neq 0\), then we have \( \chi(e) = \frac{|F(e)\Delta G(e)|}{|F(e) \cup G(e)|} \).

Let \( |F(e)| = m \), \( |G(e)| = n \), \( |H(e)| = p \), \( m \leq n \leq p \). Since \( (F,A) \subseteq (G,B) \subseteq (H,C) \), \( \forall e \in A \), then

\[
\frac{|F(e)\Delta G(e)|}{|F(e) \cup G(e)|} = \frac{n-m}{n}, \quad \frac{|F(e)\Delta H(e)|}{|F(e) \cup H(e)|} = \frac{p-m}{p}.
\]

By \( p-m/p-n-m/n = m(p-n)/pn \geq 0 \), and \( |\Delta C| \geq |\Delta B| \) we have

\[
|AB| + \sum_{e \in A \cap B} \frac{|F(e)\Delta G(e)|}{|F(e) \cup G(e)|} \leq |\Delta C| + \sum_{e \in A \cap C} \frac{|F(e)\Delta H(e)|}{|F(e) \cup H(e)|}.
\]

As a result, we get \( d((F,A),(G,B)) \leq d((F,A),(H,C)) \). Similarly we can get

\[
d((G,B),(H,C)) \leq d((F,A),(H,C)) \).
\]

When \(|F(e) \cup G(e)| = 0\), the proof is trivial.

This completes the proof of Theorem 14.

The normalized distance can be defined as:

\[
d^*(((F,A),(G,B))) = \frac{(|AB| + \sum_{e \in A \cap B} \chi(e))}{|A \cup B|}.
\]

**Example 2.** Let \((U,E)\) be a soft space with \( U = \{x_1,x_2,x_3,x_4\} \) and \( E = \{e_1,e_2,\\ e_3,e_4\} \). Suppose \((F,A)\) and \((G,B)\) are two soft sets in soft space \((U,E)\) given by:
4.2 New similarity measure

It is obvious that a soft set is determined by both parameter set and a mapping on it. So similarity measures for soft sets reflect not only the values of the same parameters, but also related the mapping about the parameters. Based upon this idea, we proposed new similarity measure in the following way:

**Theorem 15.** $S_t$ is a similarity measure, where

$$S_t((F,A),(G,B)) = \frac{|A \cap B|}{|A \cup B|} \times t,$$

$$t = \frac{\sum_{E \in A \cap B} |F(E) \cap G(E)|}{\sum_{E \in A \cup B} |F(E) \cup G(E)|}$$

if $\sum_{E \in A \cup B} |F(E) \cup G(E)| \neq 0$, and $t = 1$ otherwise.

**Proof.** (1) $\forall E \in A \cap B$, since $|F(E) \cap G(E)| \leq |F(E) \cup G(E)|$, hence

$$0 \leq S_t((F,A),(G,B)) = \frac{|A \cap B|}{|A \cup B|} \times \frac{\sum_{E \in A \cap B} |F(E) \cap G(E)|}{\sum_{E \in A \cup B} |F(E) \cup G(E)|} \leq 1 \times 1 = 1.$$

(2) If $(F,A) = (G,B)$, then $|F(E) \cap G(E)| = |F(E) \cup G(E)|$, hence

$$S_t((F,A),(F,A)) = \frac{|A \cap A|}{|A \cup A|} \times \frac{\sum_{E \in A \cap A} |F(E) \cap F(E)|}{\sum_{E \in A \cup A} |F(E) \cup F(E)|} = 1 \times 1 = 1.$$

(3) It is obvious that

$$S_t((F,A),(G,B)) = \frac{|A \cap B|}{|A \cup B|} \times \frac{\sum_{E \in A \cap B} |F(E) \cap G(E)|}{\sum_{E \in A \cup B} |F(E) \cup G(E)|} = S_t((G,B),(F,A)).$$

(4) Assume that $(F,A) \preceq (G,B) \preceq (H,C)$. Thus $A \subseteq B \subseteq C$ and for every $E \in A$, $F(E) \subseteq G(E) \subseteq H(E)$. This implies
Similarly, we can get \( S_1((F, A), (H, C)) \leq S_1((G, B), (H, C)) \).

This completes the proof of Theorem 15. Similarly, we have the following theorem.

**Theorem 16.** \( S_2 \) is a similarity measure, where

\[
S_2((F, A), (G, B)) = \frac{|A \cap B|}{|A|} \times \frac{|A \cap B|}{|B|} \times t,
\]

\[
t = \frac{\sum_{e \in A \cap B} |F(e) \cap G(e)|}{\sum_{e \in A \cap B} |F(e) \cup G(e)|} \quad \text{if} \quad \sum_{e \in A \cap B} |F(e) \cup G(e)| \neq 0, \quad \text{and} \quad t = 1 \quad \text{otherwise}.
\]

**Example 3** [3]. Profile 1. The firm \( ABC \) maintains a bearish future outlook as well as same behaviour in trading of its share prices. During last fiscal year the profit-earning ratio continued to rise. Inflation is increasing continuously \( ABC \) has a low amount of paid-up capital and a similar situation is seen in foreign direct investment flowing into \( ABC \).

Profile 2. The firm \( XYZ \) showed a fluctuating share price and hence a varying future outlook. Like \( ABC \) profit-earning ratio remained bearish. As both firms are in the same economy, inflation is also rising for \( XYZ \) and may be consider even high in view of \( XYZ \). Competition in the business area of \( XYZ \) is increasing. Debit level went high but the paid-up capital lowered.

For this, we first construct a model soft for liquidity-problem and the soft sets for the firm profile. Next we find the similarity measure of these soft sets. For the sake of ease in mathematical manipulation we denote the indication and labels by symbols as follows: \( i = \text{inflation} , \quad p = \text{profit earning ratio} , \quad s = \text{share price} \quad e_i = \text{fluctuating} , \quad c = \text{paid-up capital} \quad e_2 = \text{low} , \quad m = \text{competition} \quad e_3 = \text{rising} , \quad d = \text{business diversification} \quad e_4 = \text{high} , \quad o = \text{future outlook} \quad e_5 = \text{bearish} , \quad l = \text{debt level} \quad f = \text{foreign direct investment} \quad x = \text{fixed income} .

Thus we have soft space \((U, E), U = \{i, b, c, m, d, o, l, f, x\} \) and \( E = \{e_1, e_2, e_3, e_4, e_5\} \).

The soft sets of firm profile are as follows:

\[
A = \{e_2, e_3, e_4\}, \quad F(e_2) = \{s, f\}, \quad F(e_3) = \{p, i\}, \quad F(e_4) = \{o, s\}.
\]

\[
B = \{e_1, e_2, e_3, e_4, e_5\}, \quad G(e_1) = \{o, s\}, \quad G(e_2) = \{c\}, \quad G(e_3) = \{i, m\}.
\]
A standard soft set for a firm suffering from liquidity problem will be given. We can take it to be as follows:

\[ C = \{ e_1, e_2, e_4, e_5 \}, \quad H(e_1) = \{ o, s \}, \quad H(e_2) = \{ c \}, \quad H(e_4) = \{ i, l \}, \quad H(e_5) = \{ p, f \}. \]

Therefore:

\[ S_1 \left( (F, A), (H, C) \right) = 0.4 \times 0 = 0, \quad S_1 \left( (G, B), (H, C) \right) = 0.8 \times 1 = 0.8. \]

Hence we conclude that the firm XYZ is suffering from a liquidity problem as its soft set is significant similar to the standard liquidity problem, whereas the firm XYZ is likely to be suffering the same problem.

5. Conclusion

Majumdar and Samanta proposed some distance measures for soft sets with the same parameter sets. But the measure is a partial measure. Kharal introduced new distance measures and similarity measures for soft sets. In this paper, we have pointed out some unreasonable propositions in [3] and introduced proper results. Meanwhile, we proposed some distance and similarity measures for soft sets. And application example about soft sets is also given. We believe these approaches can help us to handle uncertain problems.

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References


On Fuzzy Upper and Lower $\beta$-irresolute Multifunctions

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Abstract:
The main purpose of this paper is to introduce and study fuzzy upper and fuzzy lower $\beta$-irresolute, $\beta$-continuous and strongly semi $\beta$-irresolute multifunctions. Also, several characterizations and properties of these multifunctions along with their mutual relationships are established in fuzzy topological spaces.

Key words and phrases:
Fuzzy topology, fuzzy multifunction, graph multifunction, upper and lower $\beta$-continuous, $\beta$-irresolute, strongly semi $\beta$-irresolute, composition, union and compactness.

1. Introduction and preliminaries

Kubiak [15] and Sostak [24] introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and fuzzy topology [6], in the sense that not only the objects are fuzzified, but also the axiomatics. In [25,26], Sostak gave some rules and showed how such an extension can be realized. Chat-topadhyay et al., [7] have redefined the same concept under the name gradation of openness. A general approach to the study of topological type structures on fuzzy power sets was developed in [10-16].

Berge [5] introduced the concept multimapping $F : X \rightarrow Y$ where $X$ and $Y$ are topological spaces and Popa [20,21] introduced the notion of irresolute multimapping. After Chang introduced the concept of fuzzy topology [6], continuity of multifunctions in fuzzy topological spaces have been defined and studied by many authors from different view points (e.g. see [3,4,17-19]). Later, Tsiporkova et al., [27, 28] introduced the Continuity of fuzzy multivalued mappings in the Chang’s fuzzy topology [6].
Throughout this paper, nonempty set will be denoted by $X$, $Y$ etc., $L = [0,1]$ and $L_0 = (0,1]$. The family of all fuzzy sets in $X$ is denoted by $L^X$. For $\alpha \in L$, $\alpha(x) = \alpha$ for all $x \in X$. A fuzzy point $x_t$ for $t \in L_0$ is an element of $L^X$ such that

$$x_t(y) = \begin{cases} t, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

The family of all fuzzy points in $X$ is denoted by $Pt(X)$. A fuzzy point $x_t \in \lambda$ iff $t < \lambda(x)$. All other notations are standard notations of fuzzy set theory.

**Definition 1.1** [2]. Let $F : X \rightarrow Y$. Then $F$ is called a fuzzy multifunction (FM, for short) iff $F(x) \in L^Y$ for each $x \in X$. The degree of membership of $y$ in $F(x)$ is denoted by $G_F(x,y) = G_F(x,y)$ for any $(x,y) \in X \times Y$.

The domain of $F$, denoted by $dom(F)$ and the range of $F$, denoted by $rng(F)$, for any $x \in X$, $y \in Y$:

$$dom(F)(x) = \bigvee_{y \in Y} G_F(x,y) \quad \text{and} \quad rng(F)(y) = \bigvee_{x \in X} G_F(x,y).$$

**Definition 1.2** [2]. Let $F : X \rightarrow Y$ be a FM. Then $F$ is called:

1. Normalized iff for each $x \in X$, there exists $y_0 \in Y$ such that $G_F(x,y_0) = 1$.
2. A crisp iff $G_F(x,y) = 1$ for each $x \in X$ and $y \in Y$.

**Definition 1.3** [2]. Let $F : X \rightarrow Y$ be a FM. Then,

1. The image of $\lambda \in L^X$ is a fuzzy set $F(\lambda) \in L^Y$ and defined by:

$$F(\lambda)(y) = \bigvee_{x \in X} \left[ G_F(x,y) \land \lambda(x) \right].$$

2. The lower inverse of $\mu \in L^Y$ is a fuzzy set $F^l(\mu) \in L^X$ and defined by:

$$F^l(\mu)(x) = \bigvee_{y \in Y} \left[ G_F(x,y) \land \mu(y) \right].$$

3. The upper inverse of $\mu \in L^Y$ is a fuzzy set $F^u(\mu) \in L^X$ and defined by:

$$F^u(\mu)(x) = \bigwedge_{y \in Y} \left[ G_F(x,y) \lor \mu(y) \right].$$

**Theorem 1.4** [2]. Let $F : X \rightarrow Y$ be a FM. Then,

1. $F(\lambda_1) \leq F(\lambda_2)$ if $\lambda_1 \leq \lambda_2$.
2. $F^l(\mu_1) \leq F^l(\mu_2)$ and $F^u(\mu_1) \leq F^u(\mu_2)$ if $\mu_1 \leq \mu_2$.
3. $F^l(1 - \mu) = 1 - F^u(\mu)$.
4. $F^u(1 - \mu) = 1 - F^u(\mu)$.
5. $F(F^u(\mu)) \leq \mu$ if $F$ is a crisp.
6. $F^u(F(\lambda)) \geq \lambda$ if $F$ is a crisp.
Definition 1.5 [2]. Let $F : X \to Y$ and $H : Y \to Z$ be two FM’s. Then the composition $H \circ F$ is defined by: $((H \circ F)(x))(z) = \bigvee_{y \in Y} [G_F(x,y) \land G_H(y,z)]$.

Theorem 1.6 [2]. Let $F : X \to Y$ and $H : Y \to Z$ be two FM’s. Then we have the following:

1. $(H \circ F) = F(H)$.
2. $(H \circ F)^u = F^u(H^u)$.
3. $(H \circ F)^l = F^l(H^l)$.

Theorem 1.7 [2]. Let $F_i : X \to Y$ be a FM. Then,

1. $(\bigcup_{i \in I} F_i)(\lambda) = \bigvee_{i \in I} F_i(\lambda)$.
2. $(\bigcap_{i \in I} F_i)^l(\mu) = \bigwedge_{i \in I} F_i^l(\mu)$.
3. $(\bigcap_{i \in I} F_i)^u(\mu) = \bigwedge_{i \in I} F_i^u(\mu)$.

Definition 1.8 [22, 24]. A fuzzy topological space (fts, for short) is pair $(X, \tau)$, where $X$ is a nonempty set and $\tau : L^X \to L$ is a mapping satisfying the following properties:

1. $\tau(\emptyset) = \tau(1) = 1$.
2. $\tau(\lambda \land \lambda_2) \geq \tau(\lambda_1) \land \tau(\lambda_2)$, for any $\lambda_1, \lambda_2 \in L^X$.
3. $\tau(\bigvee_{i \in I} \lambda_i) \geq \bigwedge_{i \in I} \tau(\lambda_i)$, for any $\{ \lambda_i \}_{i \in I} \subseteq L^X$.

Then $\tau$ is called a fuzzy topology on $X$.

Theorem 1.9 [8, 23]. Let $(X, \tau)$ be a fts. Then for each $\lambda \in L^X$, $r \in L$, we define operators $C_\tau$ and $I_\tau : L^X \times L \to L^X$ as follows:

$$C_\tau(\lambda, r) = \bigwedge \{ \mu \in L^X : \lambda \leq \mu, \tau(1 - \mu) \geq r \}.$$  
$$I_\tau(\lambda, r) = \bigvee \{ \mu \in L^X : \mu \leq \lambda, \tau(\mu) \geq r \}.$$  

For $\lambda, \mu \in L^X$ and $r, s \in L$, the operator $C_\tau$ satisfies the following statements:

1. $C_\tau(\emptyset, r) = 0$.
2. $\lambda \leq C_\tau(\lambda, r)$.
3. $C_\tau(\lambda, r) \lor C_\tau(\mu, r) = C_\tau(\lambda \lor \mu, r)$.
4. $C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.
5. $C_\tau(\lambda, r) = \lambda$ iff $\tau(1 - \lambda) \geq r$.
6. $C_\tau(1 - \lambda, r) = 1 - C_\tau(\lambda, r)$ and $I_\tau(1 - \lambda, r) = 1 - C_\tau(\lambda, r)$.

Definition 1.10 [13, 23]. Let $(X, \tau)$ be a fts. Then for each $\lambda, \mu \in L^X$ and $r \in L$, then $\lambda$ is called:
(1) \( r \)-fuzzy semi-open (\( r\)-fso, for short) iff \( \lambda \leq C_I (I_r, (\lambda, r), r) \).

(2) \( r \)-fuzzy \( \alpha \)-open (\( r\)-fao, for short) iff \( \lambda \leq I_r (C_I (I_r, (\lambda, r), r), r) \).

(3) \( r \)-fuzzy \( \beta \)-open (\( r\)-fbo, for short) iff \( \lambda \leq C_I (I_r (C_I (\lambda, r), r), r) \).

(4) \( r \)-fuzzy semi-closed (\( r\)-sc, for short) iff \( I_r (C_I (\lambda, r), r) \leq \lambda \).

(5) \( r \)-fuzzy \( \alpha \)-closed (\( r\)-fao, for short) iff \( C_I (I_r (C_I (\lambda, r), r), r) \leq \lambda \).

(6) \( r \)-fuzzy \( \beta \)-closed (\( r\)-fbo, for short) iff \( I_r (C_I (\lambda, r), r) \leq \lambda \).

**Theorem 1.11** [13]. Let \((X, \tau)\) be a fts. Then for each \( \lambda \in L^X \), \( r \in L_\tau \) we define operators \( SC_I \) and \( SI_I : L^X \times L_\tau \rightarrow L^X \) as follows:

\[
SC_I (\lambda, r) = \bigwedge \{ \mu \in L^X : \lambda \leq \mu, \mu \text{ is } r\text{-fsc} \}
\]

\[
SI_I (\lambda, r) = \bigvee \{ \mu \in L^X : \mu \leq \lambda, \mu \text{ is } r\text{-fso} \}.
\]

**Theorem 1.12** [1]. Let \((X, \tau)\) be a fts. Then for each \( \lambda \in L^X \), \( r \in L_\tau \) we define operators \( \beta C_I \) and \( \beta I_I : L^X \times L_\tau \rightarrow L^X \) as follows:

\[
\beta C_I (\lambda, r) = \bigwedge \{ \mu \in L^X : \lambda \leq \mu, \mu \text{ is } r\text{-fbo} \}.
\]

\[
\beta I_I (\lambda, r) = \bigvee \{ \mu \in L^X : \mu \leq \lambda, \mu \text{ is } r\text{-fao} \}.
\]

For \( \lambda, \mu \in L^X \) and \( r, s \in L_\tau \) the operator \( \beta C_I \) satisfies the following statements:

(1) \( \beta C_I (0, r) = 0 \).

(2) \( \lambda \leq \beta C_I (\lambda, r) \).

(3) \( \beta C_I (\lambda, r) \lor \beta C_I (\mu, r) \leq \beta C_I (\lambda \lor \mu, r) \).

(4) \( \beta C_I (\beta C_I (\lambda, r), r) = \beta C_I (\lambda, r) \).

(5) \( \lambda \) is \( r\)-fbo iff \( \beta C_I (\lambda, r) = \lambda \).

(6) \( \beta C_I (1 - \lambda, r) = 1 - \beta I_I (\lambda, r) \) and \( \beta I_I (1 - \lambda, r) = 1 - \beta C_I (\lambda, r) \).

**Definition 1.13** [9]. Let \( F : X \rightarrow Y \) be a \( FM \) between two fts’s \((X, \tau), (Y, \eta)\) and \( r \in L_\tau \). Then \( F \) is called:

(1) \( FU \alpha \) -continuous at \( x_i \in dom(F) \) iff \( x_i \in F^\alpha (\mu) \) for each \( \mu \in L^Y \) and \( \eta(\mu) \geq r \) there exists \( \lambda \in L^X \), \( \lambda \) is \( r\)-fao and \( x_i \in \lambda \) such that \( \lambda \land dom(F) \leq F^\alpha (\mu) \).

(2) \( FL \alpha \) -continuous at \( x_i \in dom(F) \) iff \( x_i \in F^l (\mu) \) for each \( \mu \in L^Y \) and \( \eta(\mu) \geq r \) there exists \( \lambda \in L^X \), \( \lambda \) is \( r\)-fao and \( x_i \in \lambda \) such that \( \lambda \leq F^l (\mu) \).

(3) \( FU \alpha \) -continuous (resp. \( FL \alpha \) -continuous) iff it is \( FU \alpha \) -continuous (resp. \( FL \alpha \) -continuous) at every \( x_i \in dom(F) \).
Theorem 1.14 [2,9]. Let $F : X \to Y$ be a FM between two fts's $(X, \tau)$, $(Y, \eta)$ and $\mu \in L^Y$. Then we have the following:

1. $F$ is FLS-continuous iff $\tau(F^u(\mu)) \geq \eta(\mu)$.
2. If $F$ is normalized, then $F$ is FUS-continuous iff $\tau(F^u(\mu)) \geq \eta(\mu)$.
3. $F$ is FL$\alpha$-continuous iff $\forall \lambda \in L$, there exists $x \in \text{dom}(F)$ such that $\lambda \cap \text{dom}(F) \subseteq F^u(\lambda)$.
4. If $F$ is normalized, then $F$ is FU$\alpha$-continuous iff $\forall \lambda \in L$, there exists $x \in \text{dom}(F)$ such that $\lambda \subseteq F^u(\lambda)$.

Definition 1.15 [14]. A fuzzy set $\lambda$ in a fts $(X, \tau)$ is called $r$-fuzzy compact iff every family in $\{ \mu : \tau(\mu) \geq r, \mu \in L^X \}$ covering $\lambda$ has a finite subcover.

Definition 1.16. A fuzzy set $\lambda$ in a fts $(X, \tau)$ is called $r$-fuzzy $\beta$-compact iff every family in $\{ \mu : \mu \in r-f\beta \lambda, \mu \in L^X \}$ covering $\lambda$ has a finite subcover.

Definition 1.17 [2]. Let $F : X \to Y$ be a FM between two fts’s $(X, \tau)$, $(Y, \eta)$ and $\mu \in L^X$. Then $F$ is called compact-valued iff $F(x)$ is r-fuzzy compact for each $x \in \text{dom}(F)$.

Definition 1.18. Let $F : X \to Y$ be a FM between two fts’s $(X, \tau)$, $(Y, \eta)$ and $\mu \in L^X$. Then $F$ is called $\beta$-compact-valued iff $F(x)$ is r-fuzzy $\beta$-compact for each $x \in \text{dom}(F)$.

2. Fuzzy upper and lower $\beta$-irresolute multifunctions

Definition 2.1. Let $F : X \to Y$ be a FM between two fts’s $(X, \tau)$, $(Y, \eta)$ and $\mu \in L^X$. Then $F$ is called:

1. Fuzzy upper $\beta$-irresolute (FU$\beta$-irresolute, for short) at a fuzzy point $x_i \in \text{dom}(F)$ iff $x_i \in F^u(\mu)$ for each $\mu \in L^Y$ and $\mu$ is r-f$\beta \lambda$ there exists $x_i \in \lambda$ such that $\lambda \cap \text{dom}(F) \subseteq F^u(\lambda)$.
2. Fuzzy lower $\beta$-irresolute (FL$\beta$-irresolute, for short) at a fuzzy point $x_i \in \text{dom}(F)$ iff $x_i \in F^l(\mu)$ for each $\mu \in L^Y$ and $\mu$ is r-f$\beta \lambda$ there exists $x_i \in \lambda$, $\lambda$ is r-f$\beta \lambda$ and $x_i \in \lambda$ such that $\lambda \cap \text{dom}(F) \subseteq F^l(\lambda)$.
3. FU$\beta$-irresolute (resp. FL$\beta$-irresolute iff it is FU$\beta$-irresolute (resp. FL$\beta$-irresolute) at every $x_i \in \text{dom}(F)$.
Definition 2.2. Let \( F : X \rightarrow Y \) be a FM between two fts’s \((X, \tau), (Y, \eta)\) and \( r \in L_{\gamma} \). Then \( F \) is called:

1. Fuzzy upper \( \beta \)-continuous (\( FU\beta \)-continuous, for short) at a fuzzy point \( x_i \in \text{dom}(F) \) iff \( x_i \in F^u(\mu) \) for each \( \mu \in L^Y \) and \( \eta(\mu) \geq r \) there exists \( \lambda \in L^X \), \( \lambda \) is \( r\)-\( f\)-\( \beta \)o and \( x_i \in \lambda \) such that \( \lambda \leq F^u(\mu) \).

2. Fuzzy lower \( \beta \)-continuous (\( FL\beta \)-continuous, for short) at a fuzzy point \( x_i \in \text{dom}(F) \) iff \( x_i \in F^l(\mu) \) for each \( \mu \in L^Y \) and \( \eta(\mu) \geq r \) there exists \( \lambda \in L^X \), \( \lambda \) is \( r\)-\( f\)-\( \beta \)o and \( x_i \in \lambda \) such that \( \lambda \leq F^l(\mu) \).

3. \( FU\beta \)-continuous (resp. \( FL\beta \)-continuous) iff it is \( FU\beta \)-continuous (resp. \( FL\beta \)-continuous) at every \( x_i \in \text{dom}(F) \).

Proposition 2.3. Let \( F : X \rightarrow Y \) be a FM between two fts’s \((X, \tau), (Y, \eta)\) and \( r \in L_{\gamma} \). If \( F \) is normalized, then \( F \) is:

1. \( FU\beta \)-irresolute at a fuzzy point \( x_i \in \text{dom}(F) \) iff \( x_i \in F^u(\mu) \) for each \( \mu \in L^Y \) and \( \mu \) is \( r\)-\( f\)-\( \beta \)o there exists \( \lambda \in L^X \), \( \lambda \) is \( r\)-\( f\)-\( \beta \)o and \( x_i \in \lambda \) such that \( \lambda \leq F^u(\mu) \).

2. \( FU\beta \)-continuous at a fuzzy point \( x_i \in \text{dom}(F) \) iff \( x_i \in F^l(\mu) \) for each \( \mu \in L^Y \) and \( \eta(\mu) \geq r \) there exists \( \lambda \in L^X \), \( \lambda \) is \( r\)-\( f\)-\( \beta \)o and \( x_i \in \lambda \) such that \( \lambda \leq F^l(\mu) \).

Theorem 2.4. Let \( F : X \rightarrow Y \) be a FM between two fts’s \((X, \tau), (Y, \eta)\) and \( \mu \in L^Y \) the following are equivalent:

1. \( F \) is \( FL\beta \)-irresolute.
2. \( F^l(\mu) \) is \( r\)-\( f\)-\( \beta \)o, for any \( \mu \) is \( r\)-\( f\)-\( \beta \)o.
3. \( F^u(\mu) \) is \( r\)-\( f\)-\( \beta \)c, for any \( \mu \) is \( r\)-\( f\)-\( \beta \)c.
4. \( \beta C_{\tau}(F^u(\mu), r) \leq F^u(\beta C_{\eta}(\mu, r)), \) for any \( \mu \in L^Y \).
5. \( I_{\tau}(\beta C_{\tau}(F^u(\mu), r), r) \leq F^u(\beta C_{\eta}(\mu, r)), \) for any \( \mu \in L^Y \).

Proof. (1) \( \Rightarrow \) (2) Let \( x_i \in \text{dom}(F), \mu \) is \( r\)-\( f\)-\( \beta \)o and \( x_i \in F^l(\mu) \) then, there exists \( \lambda \in L^X \), \( \lambda \) is \( r\)-\( f\)-\( \beta \)o and \( x_i \in \lambda \) such that \( \lambda \leq F^l(\mu) \) thus, \( x_i \in C_{\tau}(I_{\tau}(\beta C_{\tau}(F^u(\mu), r), \mu), r) \). Then, we obtain \( F^l(\mu) \leq C_{\tau}(I_{\tau}(\beta C_{\tau}(F^u(\mu), r), \mu), \mu) \). Hence, \( F^l(\mu) \) is \( r\)-\( f\)-\( \beta \)o.

(2) \( \Rightarrow \) (3) Let \( \mu \) is \( r\)-\( f\)-\( \beta \)c hence by (2), \( F^l(1-\mu) = 1-F^u(\mu) \) is \( r\)-\( f\)-\( \beta \)o. Then, \( F^u(\mu) \) is \( r\)-\( f\)-\( \beta \)c.

(3) \( \Rightarrow \) (4) Let \( \mu \in L^Y \) hence by (3), \( F^u(\beta C_{\eta}(\mu, r)) \) is \( r\)-\( f\)-\( \beta \)c. Then, we obtain \( \beta C_{\tau}(F^u(\mu), r) \leq F^u(\beta C_{\eta}(\mu, r)) \).
On Fuzzy Upper and Lower $\beta$ -irresolute Multifunctions

(4) $\Rightarrow$ (5) Let $\mu \in L^r$ hence by (4), we obtain $I_x \left( C_x \left( F^u (\mu), r \right), r \right) \leq \beta C_x \left( F^u (\mu), r \right) \leq F^u \left( \beta C_x (\mu, r) \right)$

(5) $\Rightarrow$ (2) Let $\mu$ is $r$-$f \beta o$ hence by (5), we have $1 - F^i (\mu) = F^u (1 - \mu) \geq I_x \left( C_x \left( F^u \left( 1 - \mu \right), r \right), r \right) = I_x \left( C_x \left( 1 - F^i (\mu), r \right), r \right) = 1 - C_x \left( I_x \left( C_x \left( F^i (\mu), r \right), r \right) \right)$. Then, we obtain $F^i (\mu) \leq C_x \left( I_x \left( C_x \left( F^i (\mu), r \right), r \right) \right)$ and hence $F^i (\mu)$ is $r$-$f \beta o$.

(2) $\Rightarrow$ (1) Let $x, \in \text{dom}(F), \mu$ is $r$-$f \beta o$ and $x, \in F^i (\mu)$ we have by (2), $F^i (\mu) = \lambda$ (say) is $r$-$f \beta o$ then, there exists $\lambda$ is $r$-$f \beta o$ and $x, \in \lambda$ such that $\lambda \leq F^i (\mu)$. Thus $F$ is $FL \beta$-irresolute.

**Theorem 2.5.** Let $F : X \to Y$ be a FM between two fts’s $(X, \tau), (Y, \eta)$ and $\mu \in L^r$ the following are equivalent:

1. $F$ is $FL \beta$-continuous.
2. $F^i (\mu)$ is $r$-$f \beta o$, for any $\eta(\mu) \geq r$.
3. $F^u (\mu)$ is $r$-$f \beta c$, for any $\eta(1 - \mu) \geq r$.
4. $\beta C_x \left( F^u (\mu), r \right) \leq F^u \left( C_x (\mu, r) \right)$, for any $\mu \in L^r$.
5. $I_x \left( C_x \left( F^u (\mu), r \right), r \right) \leq F^u \left( C_x (\mu, r) \right)$, for any $\mu \in L^r$.

**Proof.** (1) $\Rightarrow$ (2) Let $x, \in \text{dom}(F), \mu \in L^r$, $\eta(\mu) \geq r$ and $x, \in F^i (\mu)$ then, there exists $\lambda \in L^x$, $\lambda$ is $r$-$f \beta o$ and $x, \in \lambda$ such that $\lambda \leq F^i (\mu)$ and hence $x, \in C_x \left( I_x \left( C_x \left( F^i (\mu), r \right), r \right) \right)$. Thus, we obtain $F^i (\mu) \leq C_x \left( I_x \left( C_x \left( F^i (\mu), r \right), r \right) \right)$. Then, $F^i (\mu)$ is $r$-$f \beta o$.

(2) $\Rightarrow$ (3) Let $\mu \in L^r$ and $\eta(1 - \mu) \geq r$ hence by (2), $F^i (1 - \mu) = 1 - F^u (\mu)$ is $r$-$f \beta o$. Then, $F^u (\mu)$ is $r$-$f \beta c$.

(3) $\Rightarrow$ (4) Let $\mu \in L^r$ hence by (3), $F^u \left( C_x (\mu, r) \right)$ is $r$-$f \beta c$. Then, we obtain $\beta C_x \left( F^u (\mu), r \right) \leq F^u \left( C_x (\mu, r) \right)$.

(4) $\Rightarrow$ (5) Let $\mu \in L^r$ hence by (4), we obtain $I_x \left( C_x \left( F^u (\mu), r \right), r \right) \leq \beta C_x \left( F^u (\mu), r \right) \leq F^u \left( C_x (\mu, r) \right)$.

(5) $\Rightarrow$ (2) Let $\mu \in L^r$, $\eta(\mu) \geq r$ hence by (5), we have $1 - F^i (\mu) = F^u (1 - \mu) \geq I_x \left( C_x \left( F^u \left( 1 - \mu \right), r \right), r \right) = I_x \left( C_x \left( 1 - F^i (\mu), r \right), r \right) = 1 - C_x \left( I_x \left( C_x \left( F^i (\mu), r \right), r \right) \right)$. Then, we obtain $F^i (\mu) \leq C_x \left( I_x \left( C_x \left( F^i (\mu), r \right), r \right) \right)$ and hence $F^i (\mu)$ is $r$-$f \beta o$. 


(2) ⇒ (1) Let \( x_i \in \text{dom}(F), \mu \in \mathcal{L}, \eta(\mu) \geq r \) and \( x_i \in F^i(\mu) \) we have by (2), \( F^i(\mu) = \lambda \) (say) is \( r\beta \beta_0 \) then, there exists \( \lambda \) is \( r\beta \beta_0 \) and \( x_i \in \lambda \) such that \( \lambda \leq F^i(\mu) \). Thus \( F \) is \( FL\beta \)-continuous.

**Theorem 2.6.** Let \( F : X \rightarrow Y \) be a FM and normalized between two fts’s \((X, \tau), (Y, \eta)\) and \( \mu \in \mathcal{L} \) the following are equivalent:

1. \( F \) is \( FU\beta \)-irresolute.
2. \( F^* (\mu) \) is \( r\beta \beta_0 \), for any \( \mu \) is \( r\beta \beta_0 \) set.
3. \( F^i (\mu) \) is \( r\beta \beta \), for any \( \mu \) is \( r\beta \beta \) set.
4. \( \beta C^i_r \left( F^i(\mu), r \right) \leq F^i \left( \beta C^i_{\eta}(\mu, r) \right) \), for any \( \mu \in \mathcal{L} \).
5. \( I_\tau \left( C^i_r \left( I_\tau \left( F^i(\mu), r \right), r \right), r \right) \leq F^i \left( \beta C^i_{\eta}(\mu, r) \right) \), for any \( \mu \in \mathcal{L} \).

**Proof.** This can be proved in a similar way as Theorem 2.4.

**Theorem 2.7.** Let \( F : X \rightarrow Y \) be a FM and normalized between two fts’s \((X, \tau), (Y, \eta)\) and \( \mu \in \mathcal{L} \) the following are equivalent:

1. \( F \) is \( FU\beta \)-continuous.
2. \( F^* (\mu) \) is \( r\beta \beta_0 \), for any \( \eta(\mu) \geq r \).
3. \( F^i (\mu) \) is \( r\beta \beta \), for any \( \eta(\overline{\mu}) \geq r \).
4. \( \beta C^i_r \left( F^i(\mu), r \right) \leq F^i \left( \beta C^i_{\eta}(\mu, r) \right) \), for any \( \mu \in \mathcal{L} \).
5. \( I_\tau \left( C^i_r \left( I_\tau \left( F^i(\mu), r \right), r \right), r \right) \leq F^i \left( \beta C^i_{\eta}(\mu, r) \right) \), for any \( \mu \in \mathcal{L} \).

**Proof.** This can be proved in a similar way as Theorem 2.5.

**Corollary 2.8.** Let \( F : X \rightarrow Y \) be a FM between two fts’s \((X, \tau), (Y, \eta)\). Then

1. If \( F \) is normalized, then \( F \) is \( FU\beta \)-irresolute (resp. \( FU\beta \)-continuous) at \( x_i \) iff \( x_i \in C_\tau \left( I_\tau \left( C^i_r \left( F^i(\mu), r \right), r \right), r \right) \) for each \( \mu \) is \( r\beta \beta_0 \) (resp. \( \eta(\mu) \geq r \)) and \( x_i \in F^* (\mu) \).
2. \( F \) is \( FL\beta \)-irresolute (resp. \( FL\beta \)-continuous) at a fuzzy point \( x_i \) iff \( x_i \in C_\tau \left( I_\tau \left( F^i(\mu), r \right), r \right) \) for each \( \mu \) is \( r\beta \beta_0 \) (resp. \( \eta(\mu) \geq r \)) and \( x_i \in F^i (\mu) \).

The following implications hold:

1. \( FU\alpha \)-continuous ⇒ \( FU\beta \)-continuous ⇐ \( FU\beta \)-irresolute.
2. \( FL\alpha \)-continuous ⇒ \( FL\beta \)-continuous ⇐ \( FL\beta \)-irresolute.

In general the converses are not true.
Example 2.9. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ and $F : X \rightarrow Y$ be a FM defined by $G_F(x_1, y_1) = 0.2$, $G_F(x_1, y_2) = 1$, $G_F(x_1, y_3) = 0.3$, $G_F(x_2, y_1) = 0.5$, $G_F(x_2, y_2) = 0.3$ and $G_F(x_2, y_3) = 1$. Define fuzzy topologies $\tau : L^X \rightarrow L$ and $\eta : L^Y \rightarrow L$ as follows:

$$
\tau(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in \{0, 1\}, \\
1/2, & \text{if } \lambda = 0.5, \\
0, & \text{otherwise},
\end{cases}
$$

$$
\eta(\mu) = \begin{cases} 
1, & \text{if } \mu \in \{0, 1\}, \\
1/2, & \text{if } \mu = 0.5, \\
1/3, & \text{if } \mu = 0.4, \\
0, & \text{otherwise}.
\end{cases}
$$

(1) $F$ is $FU\beta$-continuous but not $FU\alpha$-continuous because $\eta(0.4) = 1/3$ in $(Y, \eta)$ and $F^n(0.4) = 4.3$ is not $1/3-f\alpha_\alpha$.

(2) $F$ is $FL\beta$-continuous but not $FL\alpha$-continuous because $\eta(0.4) = 1/3$ in $(Y, \eta)$ and $F^n(0.4) = 4.3$ is not $1/3-f\alpha_\alpha$.

Example 2.10. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ and $F : X \rightarrow Y$ be a FM defined by $G_F(x_1, y_1) = 0.2$, $G_F(x_1, y_2) = 1$, $G_F(x_1, y_3) = 0.3$, $G_F(x_2, y_1) = 0.5$, $G_F(x_2, y_2) = 0.3$ and $G_F(x_2, y_3) = 1$. Define fuzzy topologies $\tau : L^X \rightarrow L$ and $\eta : L^Y \rightarrow L$ as follows:

$$
\tau(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in \{0, 1\}, \\
1/2, & \text{if } \lambda = 0.5, \\
1/3, & \text{if } \mu = 0.4, \\
0, & \text{otherwise},
\end{cases}
$$

$$
\eta(\mu) = \begin{cases} 
1, & \text{if } \mu \in \{0, 1\}, \\
1/2, & \text{if } \mu = 0.5, \\
1/3, & \text{if } \mu = 0.4, \\
0, & \text{otherwise}.
\end{cases}
$$

(1) $F$ is $FU\beta$-continuous but not $FU\beta$-irresolute because $0.6$ is $1/3-f\beta_\alpha$ in $(Y, \eta)$ and $F^n(0.6) = 6.3$ is not $1/3-f\beta_\alpha$.

(2) $F$ is $FL\beta$-continuous but not $FL\beta$-irresolute because $0.6$ is $1/3-f\beta_\alpha$ in $(Y, \eta)$ and $F^n(0.6) = 6.3$ is not $1/3-f\beta_\alpha$.

Theorem 2.11. Let $\{F_i\}_{i \in I}$ be a family of $FL\beta$-irresolute between two fts’s $(X, \tau)$ and $(Y, \eta)$. Then $\bigcup_{i \in I} F_i$ is $FL\beta$-irresolute.

Proof. Let $\mu \in L^Y$, then $(\bigcup_{i \in I} F_i) (\mu) = \bigvee_{i \in I} F_i (\mu)$ by Theorem 1.7 (2). Since $\{F_i\}_{i \in I}$ is a family of $FL\beta$-irresolute between two fts’s $(X, \tau)$ and $(Y, \eta)$, then $F_i (\mu)$ is $r-f\beta_\alpha$ for any $\mu$ is $r-f\beta_\alpha$. Then we have $(\bigcup_{i \in I} F_i) (\mu) = \bigvee_{i \in I} F_i (\mu)$ is $r-f\beta_\alpha$ for any $\mu$ is $r-f\beta_\alpha$. Hence $\bigcup_{i \in I} F_i$ is $FL\beta$-irresolute.

Theorem 2.12. Let $\{F_i\}_{i \in I}$ be a family of $FL\beta$-continuous between two fts’s $(X, \tau)$ and $(Y, \eta)$. Then $\bigcup_{i \in I} F_i$ is $FL\beta$-continuous.
Proof. This can be proved in a similar way as Theorem 2.11.

**Theorem 2.13.** Let $F : X \rightarrow Y$ be a crisp $FU \beta$-irresolute and $\beta$-compact-valued between two fts’s $(X, \tau)$, $(Y, \eta)$. If $\lambda$ is $r$-fuzzy $\beta$-compact, then $F(\lambda)$ is $r$-fuzzy $\beta$-compact.

Proof. Let $\lambda$ be $r$-fuzzy $\beta$-compact set in $X$ and $\{\gamma_i : \gamma_i \text{ is } r-f \beta \alpha, i \in \Gamma\}$ be a family covering of $F(\lambda)$ i.e., $F(\lambda) \subseteq \bigcup_{\gamma_i \in \Gamma} \gamma_i$. Since $\lambda = \bigvee_{x_i \in \lambda} x_i$, we have $F(\lambda) = F\left(\bigvee_{x_i \in \lambda} x_i\right) = \bigvee_{x_i \in \lambda} F(x_i) \subseteq \bigvee_{\gamma_i \in \Gamma} \gamma_i$. It follows that for each $x_i \in \lambda$, $F(x_i) \subseteq \bigvee_{\gamma_i \in \Gamma} \gamma_i$. Since $F$ is $\beta$-compact-valued, then there exists finite subset $\{\gamma_i : \gamma_i \text{ is } \bigwedge_{x_i \in \lambda} x_i \}$ of $\{\gamma_i : \gamma_i \text{ is } \bigvee_{\gamma_i \in \Gamma} \gamma_i\}$ such that $F(\lambda) \subseteq \bigwedge_{x_i \in \lambda} x_i$. By Theorem 1.4 (6), we have $x_i \leq F^n\left(F^\gamma\left(x_i\right)\right) \leq F^u\left(\gamma_i\right)$ and $\lambda = \bigvee_{x_i \in \lambda} x_i \leq \bigvee_{x_i \in \lambda} F^u\left(\gamma_i\right)$. Since $\gamma_i$ is $r-f \beta \alpha$ then from Theorem 2.6 (2), we have $F^u\left(\gamma_i\right)$ is $r-f \beta \alpha$. Hence $\left\{F^u\left(\gamma_i\right) : \gamma_i \text{ is } \bigwedge_{x_i \in \lambda} x_i \right\}$ is the family covering the set $\lambda$. Since $\lambda$ is $r$-fuzzy $\beta$-compact, then there exists finite index set $N$ such that $\lambda = \bigwedge_{x_i \in \lambda} x_i \leq F^u\left(\gamma_i\right)$. From Theorem 1.4 (5), we have $F(\lambda) \leq F\left(\bigvee_{n \in N} F^n\left(\gamma_i\right)\right) = \bigvee_{n \in N} F\left(\bigvee_{n \in N} F^n\left(\gamma_i\right)\right)$.

Then, $F(\lambda)$ is $r$-fuzzy $\beta$-compact.

**Theorem 2.14.** Let $F : X \rightarrow Y$ be a crisp $FU \beta$-continuous and compact-valued between two fts’s $(X, \tau)$, $(Y, \eta)$. If $\lambda$ is $r$-fuzzy $\beta$-compact, then $F(\lambda)$ is $r$-fuzzy compact.

Proof. This can be proved in a similar way as Theorem 2.13.

**Theorem 2.15.** Let $F : X \rightarrow Y$ and $H : Y \rightarrow Z$ be two FM’s and let $(X, \tau)$, $(Y, \eta)$ and $(Z, \delta)$ be three fts’s. If $F$ is $FL \beta$-irresolute and $H$ is $FL \beta$-irresolute, then $H \circ F$ is $FL \beta$-irresolute.

Proof. Let $F$ be $FL \beta$-irresolute, $H$ be $FL \beta$-irresolute and $\gamma \in I^2$, $\gamma$ is $r-f \beta \alpha$. Then from Theorem 2.4 (2) we have $(H \circ F)^\gamma(\gamma) = F^i\left(H^i(\gamma)\right)$ is $r-f \beta \alpha$. Thus $H \circ F$ is $FL \beta$-irresolute.

**Theorem 2.16.** Let $F : X \rightarrow Y$ and $H : Y \rightarrow Z$ be two FM’s and let $(X, \tau)$, $(Y, \eta)$ and $(Z, \delta)$ be three fts’s. If $F$ and $H$ are normalized, $F$ is $FU \beta$-irresolute and $H$ is $FU \beta$-irresolute, then $H \circ F$ is $FU \beta$-irresolute.

Proof. This can be proved in a similar way as Theorem 2.15.
Theorem 2.17. Let \( F : X \rightarrow Y \) and \( H : Y \rightarrow Z \) be two FM’s and let \( (X, \tau) \), \( (Y, \eta) \) and \( (Z, \delta) \) be three fts’s. If \( F \) is FL\( \beta \)- irresolute and \( H \) is FL\( \beta \)-continuous, then \( H \circ F \) is FL\( \beta \)-continuous.

Proof. Let \( F \) is FL\( \beta \)- irresolute, \( H \) is FL\( \beta \)-continuous and \( \gamma \in L^\beta \), \( \delta(\gamma) \geq r \).
Then from Theorem 2.4 (2) and Theorem 2.5 (2) we have \( (H \circ F)(\gamma) = F'(H'(\gamma)) \) is \( r \)-f\( \beta \) with \( H'(\gamma) \) is \( r \)-f\( \beta \). Thus \( H \circ F \) is FL\( \beta \)-continuous.

Theorem 2.18. Let \( F : X \rightarrow Y \) and \( H : Y \rightarrow Z \) be two FM’s and let \( (X, \tau) \), \( (Y, \eta) \) and \( (Z, \delta) \) be three fts’s. If \( F \) and \( H \) are normalized, \( F \) is FU\( \beta \)- irresolute and \( H \) is FU\( \beta \)-continuous, then \( H \circ F \) is FU\( \beta \)-continuous.

Proof. This can be proved in a similar way as Theorem 2.17.

Theorem 2.19. Let \( F : X \rightarrow Y \) and \( H : Y \rightarrow Z \) be two FM’s and let \( (X, \tau) \), \( (Y, \eta) \) and \( (Z, \delta) \) be three fts’s. If \( F \) is FL\( \beta \)-continuous and \( H \) is FLS - continuous, then \( H \circ F \) is FL\( \beta \)-continuous.

Proof. Let \( F \) is FL\( \beta \)- continuous. \( H \) is FLS -continuous and \( \gamma \in L^\beta \), \( \delta(\gamma) \geq r \).
Then from Theorem 1.14 (1) and Theorem 2.5 (2) we have \( (H \circ F)(\gamma) = F'(H'(\gamma)) \) is \( r \)-f\( \beta \) with \( H'(\gamma) \) is \( r \)-f\( \beta \). Thus \( H \circ F \) is FL\( \beta \)-continuous.

Theorem 2.20. Let \( F : X \rightarrow Y \) and \( H : Y \rightarrow Z \) be two FM’s and let \( (X, \tau) \), \( (Y, \eta) \) and \( (Z, \delta) \) be three fts’s. If \( F \) and \( H \) are normalized, \( F \) is FU\( \beta \)-continuous and \( H \) is FUS - continuous, then \( H \circ F \) is FU\( \beta \)-continuous.

Proof. This can be proved in a similar way as Theorem 2.19.

Remark 2.21 [4,29]. Let \( (X, \tau) \) and \( (Y, \eta) \) be a fts’s. The fuzzy sets of the form \( \lambda \times \mu \) with \( \tau(\lambda) \geq r \) and \( \eta(\mu) \geq r \) form a basis for the product fuzzy topology \( \tau \times \eta \) on \( X \times Y \), where for any \( (x,y) \in X \times Y \), \( (\lambda \times \mu)(x,y) = \min \{ \lambda(x), \mu(y) \} \).

Definition 2.22 [4,18]. Let \( F : X \rightarrow Y \) be a FM between two fts’s \( (X, \tau) \) and \( (Y, \eta) \).
The graph fuzzy multifunction \( G_f : X \rightarrow X \times Y \) of \( F \) is defined as \( G_f(x) = x \times F(x) \) for every \( x \in X \).

Theorem 2.23. Let \( F : X \rightarrow Y \) be a FM between two fts’s \( (X, \tau) \) and \( (Y, \eta) \). If \( G_f \) is FL\( \beta \)- continuous, then \( F \) is FL\( \beta \)-continuous.

Proof. For the fuzzy sets \( \rho \in L^x, \tau(\rho) \geq r \), \( \nu \in L^y \) and \( \eta(\nu) \geq r \) we take \( (\rho \times \nu) \)
Let \( x_i \in \text{dom}(F) \), \( \mu \in L^Y \) and \( \eta(\mu) \geq r \) with \( x_i \in F^i(\mu) \), then we have \( x_i \in G_i f (X \times \mu) \) and \( \eta(X \times \mu) \geq r \). Since \( G_i f \) is \( FL \) -continuous, it follows that there exists \( \lambda \in L^X \) is \( r^-f \beta o \) and \( x_i \in \lambda \) such that \( \lambda \leq G_i f (X \times \mu) \). From here, we obtain that \( \lambda \leq F^i(\mu) \). Thus \( F \) is \( FL \beta \) -continuous.

**Theorem 2.24.** Let \( F : X \to Y \) be a FM between two fts’s \((X, \tau)\) and \((Y, \eta)\). If \( G_i f \) is \( FU \beta \) -continuous, then \( F \) is \( FU \beta \) -continuous.

**Proof.** This can be proved in a similar way as Theorem 2.23.

**Theorem 2.25.** Let \((X, \tau)\) and \((X_i, \tau_i)\) be fts’s \((i \in I)\). If a FMF: \( X \to \Pi_{i \in I} X_i \) is an \( FL \beta \) -continuous (where \( \Pi_{i \in I} X_i \) is the product space), then \( P_i \circ F \) is an \( FL \beta \) -continuous for each \( i \in I \), where \( P_i : \Pi_{i \in I} X_i \to X_i \) is the projection multifunction which is defined by \( P_i([(x_i)]) = \{x_i\} \) for each \( i \in I \).

**Proof.** Let \( \mu_0 \in L^X \) and \( \tau_\eta(\mu_0) \geq r \). Then \( (P_\eta \circ F)^i(\mu_0) = F^i[P_\eta(\mu_0)] = F^i(\mu_0 \times \Pi_{i \in I} X_i) \). Since \( F \) is \( FL \beta \) -continuous and \( \tau_\eta(\mu_0 \times \Pi_{i \in I} X_i) \geq r \), it follows that \( F^i(\mu_0 \times \Pi_{i \in I} X_i) \) is \( r^-f \beta o \). Then \( P_i \circ F \) is an \( FL \beta \) -continuous.

**Theorem 2.26.** Let \((X, \tau)\) and \((X_i, \tau_i)\) be fts’s \((i \in I)\). If a FMF: \( X \to \Pi_{i \in I} X_i \) is an \( FU \beta \) -continuous (where \( \Pi_{i \in I} X_i \) is the product space), then \( P_i \circ F \) is an \( FU \beta \) -continuous for each \( i \in I \), where \( P_i : \Pi_{i \in I} X_i \to X_i \) is the projection multifunction which is defined by \( P_i([(x_i)]) = \{x_i\} \) for each \( i \in I \).

**Proof.** This can be proved in a similar way as Theorem 2.25.

**Theorem 2.27.** Let \((X_i, \tau_i)\), \((Y_i, \eta_i)\) be fts’s and \( F_i : X_i \to Y_i \) be a FM for each \( i \in I \). Suppose that \( F : \Pi_{i \in I} X_i \to \Pi_{i \in I} Y_i \) is defined by \( F([(x_i)]) = \Pi_{i \in I} F_i(x_i) \). If \( F \) is \( FL \beta \) -continuous, then \( F_i \) is \( FL \beta \) -continuous for each \( i \in I \).

**Proof.** Let \( \mu \in L^Y \) and \( \eta_i(\mu) \geq r \). Then \( \eta_i(\mu \times \Pi_{j \neq i} Y_j) \geq r \). Since \( F \) is \( FL \beta \) -continuous, it follows that \( F^i(\mu \times \Pi_{j \neq i} Y_j) = F^i(\mu) \times \Pi_{j \neq i} X_j \) is \( r^-f \beta o \). Consequently, we obtain that \( F^i(\mu) \) is \( r^-f \beta o \) for each \( i \in I \). Thus, \( F_i \) is \( FL \beta \) -continuous.
Theorem 2.28. Let \((X_i, \tau_i), (Y_i, \eta_i)\) be fts’s and \(F_i : X_i \to Y_i\) be a FM for each \(i \in I\). Suppose that \(F : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i\) is defined by \(F((x_i)) = \prod_{i \in I} F_i(x_i)\). If \(F\) is FU\(\beta\)-continuous, then \(F_i\) is FU\(\beta\)-continuous for each \(i \in I\).

Proof. This can be proved in a similar way as Theorem 2.27.

3. Fuzzy upper and lower strongly semi \(\beta\)-irresolute multifunctions

Definition 3.1. Let \(F : X \to Y\) be a FM between two fts’s \((X, \tau), (Y, \eta)\) and \(r \in L\). Then \(F\) is called:

1. Fuzzy upper strongly semi \(\beta\)-irresolute (FUSS\(\beta\)-irresolute, for short) at a fuzzy point \(x_i \in \text{dom}(F)\) iff \(x_i \in F^u(\mu)\) for each \(\mu \in L\) and \(\mu\) is \(r\)-f\(\beta\)o there exists \(\lambda \in L^x\), \(\lambda\) is \(r\)-fso and \(x_i \in \lambda\) such that \(\lambda \land \text{dom}(F) \leq F^u(\mu)\).

2. Fuzzy lower strongly semi \(\beta\)-irresolute (FLSS\(\beta\)-irresolute, for short) at a fuzzy point \(x_i \in \text{dom}(F)\) iff \(x_i \in F^l(\mu)\) for each \(\mu \in L\) and \(\mu\) is \(r\)-f\(\beta\)o there exists \(\lambda \in L^x\), \(\lambda\) is \(r\)-fso and \(x_i \in \lambda\) such that \(\lambda \leq F^l(\mu)\).

3. FUSS\(\beta\)-irresolute (resp. FLSS\(\beta\)-irresolute (resp. FLSS\(\beta\)-irresolute) iff it is FUSS\(\beta\)-irresolute (resp. FLSS\(\beta\)-irresolute) at every \(x_i \in \text{dom}(F)\).

Proposition 3.2. If \(F\) is normalized, then \(F\) is FUSS\(\beta\)-irresolute at a fuzzy point \(x_i \in \text{dom}(F)\) iff \(x_i \in F^u(\mu)\) for each \(\mu \in L\) and \(\mu\) is \(r\)-f\(\beta\)o there exists \(\lambda \in L^x\), \(\lambda\) is \(r\)-fso and \(x_i \in \lambda\) such that \(\lambda \leq F^u(\mu)\).

Theorem 3.3. Let \(F : X \to Y\) be a FM between two fts’s \((X, \tau), (Y, \eta)\) and \(\mu \in L\) the following are equivalent:

1. \(F\) is FLSS\(\beta\)-irresolute.
2. \(F^i(\mu)\) is \(r\)-fso, for any \(\mu\) is \(r\)-f\(\beta\)o set.
3. \(F^u(\mu)\) is \(r\)-fsc, for any \(\mu\) is \(r\)-f\(\beta\)c set.
4. \(SC^\tau\left(F^u(\mu), r\right) \subseteq F^u\left(\beta C^\eta(\mu, r)\right)\) for any \(\mu \in L\).
5. \(I^\tau\left(C^\tau\left(F^u(\mu), r\right), r\right) \subseteq F^u\left(\beta C^\eta(\mu, r)\right)\) for any \(\mu \in L\).

Proof. (1) \(\Rightarrow\) (2) Let \(x_i \in \text{dom}(F)\), \(\mu\) is \(r\)-f\(\beta\)o and \(x_i \in F^i(\mu)\) then, there exists \(\lambda \in L^x\), \(\lambda\) is \(r\)-fso and \(x_i \in \lambda\) such that \(\lambda \leq F^i(\mu)\) and hence \(x_i \in C^\tau\left(I^\tau\left(F^i(\mu), r\right), r\right)\). Thus, we obtain \(F^i(\mu) \subseteq C^\tau\left(I^\tau\left(F^i(\mu), r\right), r\right)\). Then, \(F^i(\mu)\) is \(r\)-fso.

(2) \(\Rightarrow\) (3) Let \(\mu\) is \(r\)-f\(\beta\)c hence by (2), \(F^i(1 - \mu) = 1 - F^i(\mu)\) is \(r\)-fso. Then, \(F^u(\mu)\) is \(r\)-fsc.
\( (3) \Rightarrow (4) \) Let \( \mu \in L^Y \) hence by \( (3) \), \( F^\mu(\beta C^\eta(\mu), r) \) is \( r\text{-}fsc \). Then, we obtain 
\[
SC_t(F^\mu(\mu), r) \leq F^\mu(\beta C^\eta(\mu), r).
\]

\( (4) \Rightarrow (5) \) Let \( \mu \in L^Y \) hence by \( (4) \), we obtain 
\[
I_t(C_t(F^\mu(\mu), r), r) \leq SC_t(F^\mu(\mu), r) \leq F^\mu(\beta C^\eta(\mu), r).
\]

\( (5) \Rightarrow (2) \) Let \( \mu \) is \( r\text{-}f\beta c \) hence by \( (5) \), we have 
\[
1 - F^\mu(\mu) = F^\mu(1 - \mu) \geq I_t(C_t(F^\mu(1 - \mu), r), r) = I_t(C_t(1 - F^\mu(\mu), r), r).
\]
Then, we obtain 
\[
F^\mu(\mu) \leq C_t(I_t(F^\mu(\mu), r), r) \text{ and hence } F^\mu(\mu) \text{ is } r\text{-}fso.
\]

\( (2) \Rightarrow (1) \) Let \( x_i \in \text{dom}(F) \), \( \mu \) is \( r\text{-}f\beta c \) and \( x_i \in F^\mu(\mu) \) we have by \( (2) \), \( F^\mu(\mu) = \lambda \) (say) is \( r\text{-}fso \) then, there exists \( \lambda \) is \( r\text{-}fso \) and \( x_i \in \lambda \) such that \( \lambda \leq F^\mu(\mu) \). Thus \( F \) is \( FLSS\beta \)-irresolute.

**Theorem 3.4.** Let \( F : X \to Y \) be a FM and normalized between two fts’s \((X, \tau), (Y, \eta)\) and \( \mu \in L^Y \) the following are equivalent:

1. \( F \) is \( FUSS\beta \)-irresolute.
2. \( F^\mu(\mu) \) is \( r\text{-}fso \), for any \( \mu \) is \( r\text{-}f\beta c \) set.
3. \( F^\mu(\mu) \) is \( r\text{-}fsc \), for any \( \mu \) is \( r\text{-}f\beta c \) set.
4. \( SC_t(F^\mu(\mu), r) \leq F^\mu(\beta C^\eta(\mu), r) \), for any \( \mu \in L^Y \).
5. \( I_t(C_t(F^\mu(\mu), r), r) \leq F^\mu(\beta C^\eta(\mu), r) \), for any \( \mu \in L^Y \).

**Proof.** This can be proved in a similar way as Theorem 3.3.

The following implications hold:

1. \( FUSS\beta \)-irresolute \( \Rightarrow \) \( FU\beta \)-irresolute.
2. \( FLSS\beta \)-irresolute \( \Rightarrow \) \( FL\beta \)-irresolute.

In general the converse are not true.

**Example 3.5.** Let \( X = \{x_1, x_2\}, Y = \{y_1, y_2, y_3\} \) and \( F : X \to Y \) be a FM defined by 
\[
G_\mu(x_1, y_1) = 0.2, \ G_\mu(x_1, y_2) = 1, \ G_\mu(x_1, y_3) = 0.3, \ G_\mu(x_2, y_1) = 0.5, \ G_\mu(x_2, y_2) = 0.3 \text{ and } G_\mu(x_2, y_3) = 1.
\]
Define fuzzy topologies \( \tau : L^X \to L \) and \( \eta : L^Y \to L \) as follows:
\[
\tau(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in \{0, 1\}, \\
1/2, & \text{if } \lambda = 0.4, \\
0, & \text{otherwise},
\end{cases}
\]
\[
\eta(\mu) = \begin{cases} 
1, & \text{if } \mu \in \{0, 1\}, \\
1/2, & \text{if } \mu = 0.3, \\
0, & \text{otherwise},
\end{cases}
\]

We can obtain the followings:
On Fuzzy Upper and Lower \(\beta\)-irresolute Multifunctions

(1) \(F\) is \(FU\beta\)-irresolute but not \(FUSS\beta\)-irresolute because \(0.3\) is \(1/2\)-\(f\beta c\) in \((Y, \eta)\) and \(F^c (0.3) = 0.3\) is not \(1/2\)-\(f s c\).

(2) \(F\) is \(FL\beta\)-irresolute but not \(FLSS\beta\)-irresolute because \(0.3\) is \(1/2\)-\(f\beta c\) in \((Y, \eta)\) and \(F^c (0.3) = 0.3\) is not \(1/2\)-\(f s c\).

**Theorem 3.6.** Let \(\{F_i\}_{i \in I}\) be a family of \(FLSS\beta\)-irresolute between two fts’s \((X, \tau)\) and \((Y, \eta)\). Then \(\bigcup_{i \in I} F_i\) is \(FLSS\beta\)-irresolute.

**Proof.** Let \(\mu \in L^\gamma\), then \((\bigcup_{i \in I} F_i)^\gamma (\mu) = \bigvee_{i \in I} F_i^\gamma (\mu)\) by Theorem 1.7 (2). Since \(\{F_i\}_{i \in I}\) is a family of \(FLSS\beta\)-irresolute between two fts’s \((X, \tau)\) and \((Y, \eta)\), then \(F_i^\gamma (\mu)\) is \(r-fso\) for any \(\mu\) is \(r-f\beta_0\). Then we have \((\bigcup_{i \in I} F_i)^\gamma (\mu) = \bigvee_{i \in I} F_i^\gamma (\mu)\) is \(r-fso\) for any \(\mu\) is \(r-f\beta_0\). Hence \(\bigcup_{i \in I} F_i\) is \(FLSS\beta\)-irresolute.

**Theorem 3.7.** Let \(F : X \rightarrow Y\) and \(H : Y \rightarrow Z\) be two FM’s and let \((X, \tau)\), \((Y, \eta)\) and \((Z, \delta)\) be three fts’s. If \(F\) is \(FLSS\beta\)-irresolute and \(H\) is \(FL\beta\)-irresolute, then \(H \circ F\) is \(FL\beta\)-irresolute.

**Proof.** Let \(F\) is \(FLSS\beta\)-irresolute, \(H\) is \(FL\beta\)-irresolute and \(\gamma \in L^\tau\), \(\gamma\) is \(r-f\beta_0\). Then from Theorem 2.4 (2) and Theorem 3.3 (2) we have \((H \circ F)^\gamma (\gamma) = F^\gamma (H^\gamma (\gamma))\) is \(r-fso\) with \(H^\gamma (\gamma)\) is \(r-f\beta_0\). Thus \(H \circ F\) is \(FLSS\beta\)-irresolute.

**Theorem 3.8.** Let \(F : X \rightarrow Y\) and \(H : Y \rightarrow Z\) be two FM’s and let \((X, \tau)\), \((Y, \eta)\) and \((Z, \delta)\) be three fts’s. If \(F\) and \(H\) are normalized, \(F\) is \(FUSS\beta\)-irresolute and \(H\) is \(FU\beta\)-irresolute, then \(H \circ F\) is \(FUSS\beta\)-irresolute.

**Proof.** This can be proved in a similar way as Theorem 3.7.

**Theorem 3.9.** Let \(F : X \rightarrow Y\) and \(H : Y \rightarrow Z\) be two FM’s and let \((X, \tau)\), \((Y, \eta)\) and \((Z, \delta)\) be three fts’s. If \(F\) is \(FL\beta\)-irresolute and \(H\) is \(FLSS\beta\)-irresolute, then \(H \circ F\) is \(FL\beta\)-irresolute.
Proof. Let $F$ is $FL\beta$ -irresolute, $H$ is $FLSS\beta$ -irresolute and $\gamma \in L^H$, $\gamma$ is $r$-f$\beta o$. Then from Theorem 2.4 (2) and Theorem 3.3 (2) we have $(H \circ F)\left(\gamma\right) = F_i\left(H^i\left(\gamma\right)\right)$ is $r$-f$\beta o$ with $H^i\left(\gamma\right)$ is $r$-fso. Thus $H \circ F$ is $FL\beta$ -irresolute.

Theorem 3.10. Let $F : X \rightarrow Y$ and $H : Y \rightarrow Z$ be two $FM$ ’s and let $(X,\tau), (Y,\eta)$ and $(Z,\delta)$ be three $fis$’s. If $F$ and $H$ are normalized, $F$ is $FU\beta$ -irresolute and $H$ is $FUSS\beta$ -irresolute, then $H \circ F$ is $FU\beta$ -irresolute.

Proof. This can be proved in a similar way as Theorem 3.9.

4. Conclusion

It is well known that fuzzy set theory has been regarded as a generalization of classical set theory in one way. Furthermore, this is an important mathematical tool to deal with uncertainty. One of the main contributions of this paper is introduce the concepts of fuzzy upper and lower $\beta$ -continuous, $\beta$ -irresolute, strongly semi $\beta$ -irresolute multifunctions on fuzzy topological spaces in Sostak’s sense and investigate some of their properties. Also, the relationship between these multifunctions are investigated. Later, some applications are introduced and studied.

References


Rough Prime Bi-ideals in $\Gamma$-semigroups

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Abstract:
In this paper we introduce the notions rough prime bi-ideals, strongly rough prime bi-ideals, rough semiprime bi-ideals, strongly rough irreducible and rough irreducible bi-ideals of $\Gamma$-semigroups. We have shown that the lower and upper approximation of a prime bi-ideal are also prime bi-ideals.

Keywords:
Prime bi-ideals, Strongly rough prime bi-ideals, Irreducible bi-ideals, Rough semiprime bi-ideals, Strongly rough irreducible bi-ideals, Rough irreducible bi-ideals.

1. Introduction

The notion of a rough set was originally proposed by Pawalk [10-13] as a formal tool for modeling and processing incomplete data in information systems. In rough set theory any subset of the universal set which has uncertainty or incomplete in nature is represented by two ordinary subsets of the universe known as the lower and upper approximations of the given set. The equivalence classes are used as the building blocks for the construction of the lower and upper approximations. The lower approximation of a given set is the union of all equivalence classes which are subsets of the set and the upper approximation is union of all equivalence classes which have a nonempty intersection with the set. Some authors have studied the algebraic properties of rough sets. Aslam et.al [1], Biswas and Nanda [2], Chinram [3,4], Davvaz [5], Jun [6], Kuroki and Mordeson [7], Kuroki [8] and Thillaigovindan et. al [18,19], Xiao et. al [20] are some notable works where roughness has been applied in different algebraic structures.

The fundamental concept of $\Gamma$-semigroup was introduced by Sen in [16, 17]. The $\Gamma$-semigroup is a generalization of semigroup. Many researchers have worked on $\Gamma$-semigroup and its sub structures. Many classical notions of semi-groups have been
extended to $\Gamma$-semigroup and its sub structures. Many classical notions of semi-groups have been extended to $\Gamma$-semigroups by Sha and Sen in [14], [15] and [17]. On the other hand the concept of prime bi-ideal, strongly prime bi-ideal, semiprime bi-ideal, strongly irreducible bi-ideal and irreducible bi-ideal of $\Gamma$-semigroups are studied in [9].

In this paper we introduce the concept of rough prime bi-ideals, strongly rough prime bi-ideals, rough semiprime bi-ideals, strongly rough irreducible and rough irreducible bi-ideals of $\Gamma$-semigroups.

2. Preliminaries

In this section we reproduce some basic concepts which are needed in the sequel.

Let $U$ be a universal set. For an equivalence relation $\theta$ on $U$, the set of elements of $U$ that are related to $x \in U$, is called the equivalence class of $x$ and is denoted by $[x]_{\theta}$. Let $U/\theta$ denote the family of all equivalence classes induced by $\theta$ on $U$. $U/\theta$ is a partition of $U$ such that each element of $U$ is contained in exactly one equivalence class.

Definition 2.1. A pair $(U, \theta)$ where $U \neq \emptyset$ and $\theta$ is an equivalence relation on $U$, is called an approximation space.

Definition 2.2. For an approximation $(U, \theta)$ by a rough approximation in $(U, \theta)$ we mean a mapping $\rho : \mathcal{P}(U) \rightarrow \mathcal{P}(U) \times \mathcal{P}(U)$ defined as $\rho(X) = \left(\underline{\rho}(X), \overline{\rho}(X)\right)$ for $X \subseteq U$, where $\underline{\rho}(X) = \left\{x \in U \mid [x]_{\theta} \subseteq X\right\}$, $\overline{\rho}(X) = \left\{x \in U \mid [x]_{\theta} \cap X \neq \emptyset\right\}$. $\underline{\rho}(X)$ is called a lower rough approximation of $X$ in $(U, \theta)$ where as $\overline{\rho}(X)$ is called upper approximation of $X$ in $(U, \theta)$.

Definition 2.3. Given an approximation space $(U, \theta)$, a pair $(A, B) \in \mathcal{P}(U) \times \mathcal{P}(U)$ is called a rough set in $(U, \theta)$ if and only if $(A, B) = \rho(X)$ for some $X \subseteq U$.

Definition 2.4. Let $M$ be a $\Gamma$-semigroup. A sub $\Gamma$-semigroup $B$ of $M$ is called a bi-ideal of $M$ if $B \Gamma M \cap M \Gamma B \subseteq B$.

Definition 2.5. Let $M$ be a $\Gamma$-semigroup. An ideal $P$ of $M$ is said to be prime ideal of $M$ if for any two ideals $A$ and $B$ of $M$, $AB \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$.

Definition 2.6. Let $M$ be a $\Gamma$-semigroup. An ideal $P$ of $M$ is said to be semiprime ideal of $M$ if for any ideal $A$ of $M$, $A \Gamma A \subseteq P$ implies that $A \subseteq P$.

3. Rough prime bi-ideals in $\Gamma$-semigroups
In this section we introduce rough prime bi-ideals, strongly rough prime bi-ideals, rough semiprime bi-ideals, strongly rough irreducible and rough irreducible bi-ideals of \( \Gamma \)-semigroups. Throughout this paper \( M \) denotes a \( \Gamma \)-semigroup unless otherwise specified.

Let \( \theta \) be an equivalence relation on \( M \). \( \theta \) is called a congruence relation on \( M \), whenever \((a,b) \in \theta \) implies \((a\gamma x, b\gamma x) \in \theta \) and \((x\gamma a, x\gamma b) \in \theta \) for all \( x \in M \) and \( \gamma \in \Gamma \). We denote by \([a]_\theta\) the equivalence class containing the element \( a \in M \).

A congruence relation \( \theta \) on \( M \) is said to be complete if \([a]_\theta \Gamma [b]_\theta = [a\Gamma b]_\theta \) for all \( a, b \in M \).

**Definition 3.1.** Let \( A \) be a nonempty subset of \( M \) and \( \theta \) be a congruence relation on \( M \). Then the \( \theta \)-lower and \( \theta \)-upper approximations of \( A \) are defined by
\[
\theta^- (A) = \{ x \in M \mid \exists y \in A \text{ such that } x \theta y \}, \\
\theta^+ (A) = \{ x \in M \mid \exists y \in A \text{ such that } y \theta x \}.
\]

A nonempty subset \( A \) of \( M \) is called \( \theta \)-upper (resp. \( \theta \)-lower) rough sub \( \Gamma \)-semigroup of \( M \) if \( \theta^- (A) \) (resp. \( \theta^+ (A) \)) is a sub \( \Gamma \)-semigroup of \( M \).

**Definition 3.2.** A nonempty subset \( A \) of \( M \) is called a \( \theta \)-upper (resp. \( \theta \)-lower) rough bi-ideal of \( M \) if \( \theta^- (A) \) (resp. \( \theta^+ (A) \)) is a bi-ideal of \( M \), where \( \theta \) is a congruence relation on \( M \).

**Theorem 3.3** [18]. Let \( \theta \) and \( \Psi \) be congruence relations on \( M \) and let \( A \) and \( B \) be nonempty subsets of \( M \). Then the following are true:

(i) \( \theta (A) \subseteq A \subseteq \theta (A) \),
(ii) \( \theta (\phi) = \phi = \theta (\theta) \),
(iii) \( \theta (M) = M = \theta (M) \) and \( \theta (\Gamma) = \Gamma = \theta (\Gamma) \),
(iv) \( \theta^+ (A \cup B) = \theta^+ (A) \cup \theta^+ (B) \),
(v) \( \theta^- (A \cap B) = \theta^- (A) \cap \theta^- (B) \),
(vi) \( A \subseteq B \) implies \( \theta^- (A) \subseteq \theta^- (B) \) and \( \theta^+ (A) \subseteq \theta^+ (B) \),
(vii) \( \theta^- (A \cup B) \supseteq \theta^- (A) \cup \theta^- (B) \),
(viii) \( \theta^+ (A \cap B) \supseteq \theta^+ (A) \cap \theta^+ (B) \),
(ix) \( \theta \subseteq \Psi \) implies \( \Psi (A) \subseteq \theta (A) \) and \( \theta (A) \subseteq \Psi (A) \),
(x) \( \theta (A) \Gamma \theta (B) \subseteq \theta (A \Gamma B) \),
(xi) If \( \theta \) is complete, then \( \theta (A) \Gamma \theta (B) \subseteq \theta (A \Gamma B) \),
(xii) \( \theta^- (\Psi (A) \cap \theta (A) \cap \Psi (A) \),
(xiii) \( \theta^- (\Psi (A) \cap \theta (A) \cap \Psi (A) \),
(xiv) \( \theta (\theta (A)) = \theta (A) \),
(xv) \( \theta (\theta (A)) = \theta (A) \).
(xvi) $\bar{\theta}(\theta(A)) = \theta(A)$.
(xvii) $\theta(\bar{\theta}(A)) = \bar{\theta}(A)$.

**Theorem 3.4** [18]. Let $\theta$ be a congruence relation on $M$ and let $A$ be a bi-ideal of $M$. Then

(i) $\bar{\theta}(A)$ is a bi-ideal of $M$

(ii) If $\theta$ is complete then $\theta(A)$ is a bi-ideal of $M$.

**Theorem 3.5** [18]. Let $\theta$ be a congruence relation on $M$ and let $A$ be a left (right) ideal of $M$ and $\bar{\theta} (A)$ be nonempty, then

(i) $\bar{\theta}(A)$ is both a bi-ideal and quasi-ideal of $M$ and

(ii) $\theta(A)$ is both a bi-ideal and quasi-ideal of $M$.

**Definition 3.6.** A bi-ideal $B$ of a $\Gamma$-semigroup $M$ is called prime bi-ideal (strongly prime bi-ideal) of $M$ if $1 \subseteq B \Longleftrightarrow \theta(B) \subseteq \bar{\theta}(B)$ implies that either $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-ideals $B_1$ and $B_2$ of $M$.

**Definition 3.7.** A bi-ideal $B$ of a $\Gamma$-semigroup $M$ is called semiprime bi-ideal of $M$ if $A \subseteq B$ implies that $A \subseteq B$ for any bi-ideal $A$ of $M$.

**Definition 3.8.** Let $\theta$ be a congruence relation on $M$. A bi-ideal $B$ of $M$ is called rough prime bi-ideal (strongly rough prime bi-ideal) of $M$ if $\bar{\theta}(B)$ and $\theta(B)$ are prime bi-ideals (strongly prime bi-ideals) of $M$.

**Definition 3.9.** Let $\theta$ be a congruence relation on $M$. A bi-ideal $B$ of $M$ is called rough semiprime bi-ideal of $M$ if $\bar{\theta}(B)$ and $\theta(B)$ are semiprime bi-ideals of $M$.

**Definition 3.10.** A bi-ideal $B$ of a $\Gamma$-semigroup is called irreducible (strongly irreducible) bi-ideal of $M$ if $B_1 \cap B_2 = B \subseteq \bar{\theta}(B)$ implies that either $B_1 = \bar{\theta}(B)$ or $B_2 = \bar{\theta}(B)$ for all bi-ideals $B_1$, $B_2$ of $M$.

**Definition 3.11.** Let $\theta$ be a congruence relation on $M$. A bi-ideal $B$ of $M$ is called rough irreducible (strongly irreducible) bi-ideal of $M$ if $\bar{\theta}(B)$ and $\theta(B)$ are irreducible (strongly irreducible) bi-ideals of $M$.

**Theorem 3.12.** Let $\theta$ be a congruence relation on $M$. If $A$ is a prime bi-ideal of $M$, then $\bar{\theta}(A)$ is a prime bi-ideal of $M$.

**Proof.** Let $A$ be a prime bi-ideal of $M$. Then $A_1 \Gamma A_2 \subseteq A$ implies that either $A_1 \subseteq A$ or $A_2 \subseteq A$ for any bi-ideals $A_1$ and $A_2$ of $M$. Since $A$ be a bi-ideal of $M$. By Theorem 3.4 $\bar{\theta}(A)$ is a bi-ideal of $M$. Assume that $\bar{\theta}(A_1) \Gamma \bar{\theta}(A_2) \subseteq \bar{\theta}(A)$, $\bar{\theta}(A_1) \subseteq$
Theorem 3.13. Let $\theta$ be a congruence relation on $M$. If $A$ is a prime bi-ideal of $M$, then $\theta(A)$ is a prime bi-ideal of $M$.

Proof. The proof is similar to Theorem 3.12.

Theorem 3.14. Let $\theta$ be a complete congruence relation on $M$ and $A$ a prime bi-ideal of $M$. If $\theta(A)$ is nonempty then $\theta(A)$ is a prime bi-ideal of $M$.

Proof. By Theorem 3.12 $\theta(A)$ is a prime bi-ideal of $M$ and by Theorem 3.13 $\theta(A)$ is a prime bi-ideal of $M$. Hence $\theta(A)$ is a prime bi-ideal of $M$.

Theorem 3.15. The intersection of a family of prime bi-ideals of $M$ is a rough semiprime bi-ideal of $M$.

Proof. The intersection of family of prime bi-ideals of $M$ is a semiprime bi-ideal of $M$. Therefore it is a rough semiprime bi-ideal of $M$.

Theorem 3.16. Every strongly irreducible semiprime bi-ideal of $M$ is a strongly rough prime bi-ideal of $M$.

Proof. Let $B$ be strongly irreducible semiprime bi-ideal of $M$. Then $\theta(B)$ is strongly irreducible semiprime bi-ideal of $M$. Let $B_1$ and $B_2$ be any two bi-ideals of $M$, then $\theta(B_1)$ and $\theta(B_2)$ are bi-ideals of $M$ such that 

$$\theta(B_1) \cap \theta(B_2) \subseteq \theta(B).$$

Since $\theta(B_1) \cap \theta(B_2) \subseteq \theta(B_1) \cap \theta(B_2)$ and $\theta(B_1) \cap \theta(B_2)$ are bi-ideals of $M$, we have $\theta(B_1) \cap \theta(B_2) \subseteq \theta(B)$. Also since $\theta(B)$ is semiprime bi-ideal of $M$, we have either $\theta(B_1) \subseteq \theta(B)$ or $\theta(B_2) \subseteq \theta(B)$. Thus $\theta(B)$ is a strongly prime bi-ideal of $M$. Similarly $\theta(B)$ is also a strongly prime bi-ideal of $M$. Hence $\theta(B)$ is a strongly rough prime bi-ideals of $M$.

Lemma 3.17. Let $B$ be a bi-ideal of $M$ and $a \in M$ such that $a \not\in B$. Then there exists a rough irreducible bi-ideal $\theta(I)$ of $M$ such that $\theta(I) \subseteq \theta(B)$ and $a \not\in \theta(I)$.

Proof. Let $B$ be a bi-ideal of $M$. Then $\theta(B)$ is also a bi-ideal of $M$. Let $\mathcal{A}$ be the collection of all rough bi-ideals of $M$ which contain $\theta(B)$ and do not contain $a$. Since $\theta(B) \in \mathcal{A}$, $\mathcal{A}$ is nonempty. The collection $\mathcal{A}$ is a partially ordered set under set
inclusion. If $\beta$ is any totally ordered subset of $\mathcal{A}$, then the union of all the subset in $\beta$ is a bi-ideal of $M$ containing $\theta(B)$ and not containing $a$.

Hence by Zorn’s lemma there exists a maximal element $\theta(I)$ in $\mathcal{A}$. We show that $\theta(I)$ is an irreducible bi-ideal of $M$. Let $\theta(C)$ and $\theta(D)$ be two bi-ideals if $M$ such that $\theta(I) = \theta(C) \cap \theta(D)$. If both $\theta(C)$ and $\theta(D)$ properly contain $\theta(I)$, then $a \in \theta(C)$ and $a \in \theta(D)$. Thus $a \in (\theta(C) \cap \theta(D)) = \theta(I)$. This contradicts the fact that $a \notin \theta(I)$. Thus either $\theta(I) = \theta(C)$ or $\theta(I) = \theta(D)$. Hence $\theta(I)$ is irreducible bi-ideal. Similary $\theta(I)$ is irreducible bi-ideal. Thus $\theta(I)$ is a rough irreducible bi-idea.

**Definition 3.18.** An element $a \in M$ is called regular in a $\Gamma$-semigroup $M$ if $a \alpha M a \alpha \Gamma a = \alpha M a \alpha \Gamma a$ and $a \alpha M a \alpha x = \alpha M a \alpha x$. $M$ is called regular if every element of $M$ is regular.

**Definition 3.19.** An element $a \in M$ is called intra-regular in a $\Gamma$-semigroup $M$ if $a M a = a M a$. $M$ is called intra-regular if every element of $M$ is intra-regular.

**Theorem 3.20.** For the $\Gamma$-semigroup $M$ the following conditions are equivalent

(i) $M$ is intra-regular.

(ii) $\theta(R) \cap \theta(L) = \theta(R) \cap \theta(L)$ for every right ideal $R$ and left ideal $L$ of $M$.

**Proof.** (i) $\Rightarrow$ (ii) Suppose $M$ is intra-regular. Let $x \in \theta(R) \cap \theta(L)$. Then $x \in \theta(R)$ and $x \in \theta(L)$. Since $M$ is intra-regular for any $x \in M$, we have $x \in M \Gamma x \Gamma = (M \Gamma x) \Gamma (x \Gamma M)$. This implies that $x \in \theta(L) \Gamma \theta(R)$. Thus $\theta(R) \cap \theta(L) \subseteq \theta(L) \Gamma \theta(R)$. A similar proof holds for $\theta(R) \cap \theta(L) \subseteq \theta(L) \Gamma \theta(R)$. Thus $\theta(R) \cap \theta(L) \subseteq \theta(L) \Gamma \theta(R)$.

(ii) $\Rightarrow$ (i) Suppose $\theta(R) \cap \theta(L) \subseteq \theta(L) \Gamma \theta(R)$. Since $\theta(L)$ is a left ideal and $\theta(R)$ is a right ideal of $M$, by hypothesis we have $(\theta(R) \Gamma M) \cap (M \Gamma \theta(L)) \subseteq (M \Gamma \theta(L)) \cap (\theta(R) \Gamma M) \subseteq M \Gamma (\theta(L) \Gamma \theta(R)) \Gamma M$. Let $y \in (\theta(R) \Gamma M) \cap (M \Gamma \theta(L))$ then $y \in (\theta(R) \Gamma M)$ and $y \in (M \Gamma \theta(L))$. These imply that $y \in \theta(R)$ and $y \in \theta(L)$. Therefore $y \in M \Gamma y \Gamma M$ for all $y \in M$. Hence $M$ is intra-regular.

**Theorem 3.21.** For a $\Gamma$-semigroup $M$, the following conditions are equivalent.

(i) $M$ is both regular and intra regular.

(ii) $\theta(B) \Gamma \theta(B) = \theta(B)$ for every ideal $B$ of $M$.

(iii) $\theta(B_1) \cap \theta(B_2) = (\theta(B_1) \Gamma \theta(B_2)) \cap (\theta(B_2) \Gamma \theta(B_1))$ for all bi-ideals $\theta(B_1)$ and $\theta(B_2)$ of $M$.

(iv) Each bi-ideal of $M$ is semiprime.

(v) Each proper bi-ideal of $M$ is the intersection of irreducible semiprime bi-ideals of $M$ which contain it.
Proof. (i) ⇒ (ii) Let \( M \) be both regular and intra-regular and let \( B \) be a bi-ideal of \( M \). By Theorem 3.4 \( \overline{\Theta}(B) \) is a bi-ideal of \( M \).

\[
\overline{\Theta}(B) \Gamma \overline{\Theta}(B) \subseteq \overline{\Theta}(B \Gamma B) \subseteq \overline{\Theta}(B) \tag{1}
\]

Let \( x \in \overline{\Theta}(B) \). Since \( M \) is regular there exists \( x \in M = \overline{\Theta}(M) \) such that \( x \in x \Gamma M \Gamma x = x \Gamma M \Gamma (x \Gamma M) \Gamma x \). Since \( M \) is intra-regular, for \( x \in M \), \( x \in MG \Gamma x \Gamma M \). Thus \( x \in x \Gamma M \Gamma (MG \Gamma x) \Gamma (x \Gamma (MG \Gamma ) \Gamma x) \). These imply that \( x \in (\overline{\Theta}(B) \Gamma \overline{\Theta}(M) \Gamma \overline{\Theta}(B)) \Gamma (\overline{\Theta}(B) \Gamma \overline{\Theta}(M) \Gamma \overline{\Theta}(B)) \)

\[
\subseteq \overline{\Theta}(B \Gamma MB \Gamma B) \subseteq \overline{\Theta}(B) \Gamma \overline{\Theta}(B) \tag{2}
\]

Thus \( \overline{\Theta}(B) \Gamma \overline{\Theta}(B) = \overline{\Theta}(B) \) [by (1) and (2)]. Similarly \( \overline{\Theta}(B) \Gamma \overline{\Theta}(B) = \overline{\Theta}(B) \). Hence \( \overline{\Theta}(B) \Gamma \overline{\Theta}(B) = \overline{\Theta}(B) \)

(ii) ⇒ (iii) Let \( B_1 \) and \( B_2 \) are two bi-ideals of \( M \). Then \( \overline{\Theta}(B_1) \) and \( \overline{\Theta}(B_2) \) are bi-ideals of \( M \). By hypothesis \( \overline{\Theta}(B_1) \cap \overline{\Theta}(B_2) = (\overline{\Theta}(B_1) \cap \overline{\Theta}(B_2) \cap \overline{\Theta}(B_1)) \cap \overline{\Theta}(B_2) = (\overline{\Theta}(B_1) \cap \overline{\Theta}(B_2) \cap \overline{\Theta}(B_1)) \cap \overline{\Theta}(B_2) \). Similarly \( \overline{\Theta}(B_1) \cap \overline{\Theta}(B_2) \subseteq \overline{\Theta}(B_2) \Gamma \overline{\Theta}(B_1) \). Therefore \( \overline{\Theta}(B_1) \Gamma \overline{\Theta}(B_2) \) and \( \overline{\Theta}(B_2) \Gamma \overline{\Theta}(B_1) \) are bi-ideals, and so \( (\overline{\Theta}(B_1) \Gamma \overline{\Theta}(B_2)) \Gamma (\overline{\Theta}(B_2) \Gamma \overline{\Theta}(B_1)) \) is a bi-ideal of \( M \). Thus \( (\overline{\Theta}(B_1) \Gamma \overline{\Theta}(B_2)) \cap (\overline{\Theta}(B_1) \Gamma \overline{\Theta}(B_2)) = (\overline{\Theta}(B_1) \Gamma \overline{\Theta}(B_2)) \cap (\overline{\Theta}(B_1) \Gamma \overline{\Theta}(B_2)) \). Therefore \( (\overline{\Theta}(B_1) \Gamma \overline{\Theta}(B_2)) \cap (\overline{\Theta}(B_1) \Gamma \overline{\Theta}(B_2)) \subseteq \overline{\Theta}(B_1) \Gamma \overline{\Theta}(B_2) \Gamma \overline{\Theta}(B_2) \Gamma \overline{\Theta}(B_1) \Gamma \overline{\Theta}(B_2) \Gamma \overline{\Theta}(B_1) \Gamma \overline{\Theta}(B_2) \Gamma \overline{\Theta}(B_1) \Gamma \overline{\Theta}(B_2) \Gamma \overline{\Theta}(B_1) \)

Similarly \( (\overline{\Theta}(B_1) \Gamma \overline{\Theta}(B_2)) \cap (\overline{\Theta}(B_1) \Gamma \overline{\Theta}(B_2)) \subseteq \overline{\Theta}(B_1) \Gamma \overline{\Theta}(B_2) \). Thus \( (\overline{\Theta}(B_1) \Gamma \overline{\Theta}(B_2)) \cap (\overline{\Theta}(B_1) \Gamma \overline{\Theta}(B_2)) = \overline{\Theta}(B_1) \Gamma \overline{\Theta}(B_2) \). Hence \( \overline{\Theta}(B_1) \cap \overline{\Theta}(B_2) = \overline{\Theta}(B_1 \Gamma B_2 \cap B_2 \Gamma B_1) \).

(iii) ⇒ (iv) Let \( B_1 \) and \( B \) be any two bi-ideals of \( M \). Then \( \overline{\Theta}(B_1) \) and \( \overline{\Theta}(B) \) are also bi-ideals of \( M \) such that \( \overline{\Theta}(B_1) \Gamma \overline{\Theta}(B) \subseteq \overline{\Theta}(B) \). By hypothesis \( \overline{\Theta}(B_1) = \overline{\Theta}(B_1) \cap \overline{\Theta}(B) = \overline{\Theta}(B_1) \cap \overline{\Theta}(B) \cap \overline{\Theta}(B_1) \cap \overline{\Theta}(B_1) \cap \overline{\Theta}(B_1) \cap \overline{\Theta}(B) \). Hence \( \overline{\Theta}(B) \) is a semiprime bi-ideal of \( M \). Similarly \( \overline{\Theta}(B) \) is also a semiprime bi-ideal of \( M \). Therefore \( \overline{\Theta}(B) \) is a rough semiprime bi-ideal of \( M \).

(iv) ⇒ (v) Let \( B \) be a proper bi-ideal of \( M \), then \( \overline{\Theta}(B) \) is also a proper bi-ideal of \( M \) and \( \overline{\Theta}(B) \) is contained in the intersection of all rough irreducible bi-ideals of \( M \) which contain \( \overline{\Theta}(B) \). Lemma 3.17 guarantees the existence of such irreducible bi-ideals. If \( a \in \overline{\Theta}(B) \) then there exists an irreducible bi-ideal of \( M \) which contains \( \overline{\Theta}(B) \) but does not contain \( a \). Hence \( \overline{\Theta}(B) \) is the intersection of all \( \overline{\Theta} \)-lower rough irreducible semiprime bi-ideals of \( M \) which contain \( \overline{\Theta}(B) \). By hypothesis, every \( \overline{\Theta} \)-lower rough
bi-ideal is semiprime and so each $\theta$-lower rough bi-ideal is the intersection of $\theta$-lower rough irreducible semiprime bi-ideals of $M$ containing $\theta$-lower rough bi-ideal. Similarly every $\theta$-upper rough bi-ideal is semiprime and so each $\theta$-upper rough bi-ideal is the intersection of $\theta$-upper rough irreducible semiprime bi-ideals of $M$ containing $\theta$-upper rough bi-ideal. Hence each proper bi-ideal of $M$ is the intersection of irreducible rough semiprime bi-ideals of $M$ which containing it.

(v) $\Rightarrow$ (ii) Let $B$ be a bi-ideal of $M$. Then $\overline{\theta}(B)$ is a bi-ideal of $M$. If $\overline{\theta}(B)\Gamma \overline{\theta}(B) = \overline{\theta}(M) = M$, then clearly $\overline{\theta}(B)$ is idempotent. If $\overline{\theta}(B)\Gamma \overline{\theta}(B) \neq \overline{\theta}(M)$ then $\overline{\theta}(B)$ $\Gamma \overline{\theta}(B)$ is a proper bi-ideal of $M$ containing $\overline{\theta}(B)\Gamma \overline{\theta}(B)$. Thus by hypothesis $\overline{\theta}(B)$ $\Gamma \overline{\theta}(B) = \bigcap_{a} \left[ \overline{\theta}(B_{a}) : \overline{\theta}(B_{a}) \right]$ is an irreducible semiprime bi-ideal of $M$ containing $\overline{\theta}(B)\Gamma \overline{\theta}(B)$. Since each $\overline{\theta}(B_{a})$ is a semiprime bi-ideal of $M$, and $\overline{\theta}(B) \subseteq \overline{\theta}(B_{a})$ for all $\alpha$ implies that $\overline{\theta}(B) \subseteq \bigcap_{a} \overline{\theta}(B_{a}) = \overline{\theta}(B)\Gamma \overline{\theta}(B)$. But $\overline{\theta}(B)\Gamma \overline{\theta}(B) \subseteq \overline{\theta}(B)$ (because $\overline{\theta}(B)$ is a sub $\Gamma$-semigroup). Thus each $\theta$-upper rough bi-ideal of $M$ is idempotent. A similar proof holds for the bi-ideal $\underline{\theta}(B)$ of $M$. Hence each rough bi-ideal of $M$ is idempotent.

(ii) $\Rightarrow$ (i) Suppose that $\overline{\theta}(B)\Gamma \overline{\theta}(B)$ for all bi-ideals $\overline{\theta}(B)$ of $M$. By Theorem 3.5 $\overline{\theta}(L)\cap \overline{\theta}(R)$ is a bi-ideal of $M$ for all right ideals $R$ and left ideals $L$ of $M$. By hypothesis $\overline{\theta}(L)\cap \overline{\theta}(R) = (\overline{\theta}(L)\cap \overline{\theta}(R))\Gamma (\overline{\theta}(L)\cap \overline{\theta}(R)) \subseteq \overline{\theta}(L)\Gamma \overline{\theta}(R)$. Thus by Theorem 3.21 $M$ is intra-regular. Also $\overline{\theta}(L)\cap \overline{\theta}(R) = (\overline{\theta}(L)\cap \overline{\theta}(R))\Gamma (\overline{\theta}(L)\cap \overline{\theta}(R)) \subseteq \overline{\theta}(L)\Gamma \overline{\theta}(R)$. But $\overline{\theta}(R)\Gamma \overline{\theta}(L) \subseteq \overline{\theta}(L)\cap \overline{\theta}(R)$ always holds. Thus $\overline{\theta}(R)\Gamma \overline{\theta}(L) = \overline{\theta}(L)\overline{\theta}(R)$. Therefore $\overline{\theta}(L)\cap \overline{\theta}(R) = \overline{\theta}(R)\Gamma \overline{\theta}(L) = \overline{\theta}(R)\Gamma \overline{\theta}(L) \subseteq \overline{\theta}(R)\Gamma M\Gamma \overline{\theta}(L) \subseteq \overline{\theta}(R)\Gamma M\Gamma \overline{\theta}(L)$. Let $x \in \overline{\theta}(L)\cap \overline{\theta}(R)$, then $x \in \overline{\theta}(L)$ and $x \in \overline{\theta}(R)$. Thus $x \in x\Gamma M\Gamma x$ for all $x \in M$. Hence $M$ is regular.

**Theorem 3.22.** Let $M$ be regular and intra-regular. Then the following are equivalent for the bi-ideal $B$ of $M$

(i) $\underline{\theta}(B)$ is strongly rough irreducible.

(ii) $\overline{\theta}(B)$ is strongly rough prime

**Proof.** (i) $\Rightarrow$ (ii) Let $M$ be regular and intra-regular and $B$ be a bi-ideal of $M$. Then $\overline{\theta}(B)$ is bi-ideal of $M$. By Theorem 3.21, $\overline{\theta}(B)$ is semiprime. Since $\overline{\theta}(B)$ is strongly irreducible, by Lemma 3.17, $\overline{\theta}(B)$ is strongly prime bi-ideal of $M$. The proof of $\underline{\theta}(B)$ is strongly prime bi-ideal of $M$ is similar. Thus $\overline{\theta}(B)$ is strongly rough prime bi-ideal of $M$.

(ii) $\Rightarrow$ (i) Let $B$ be strongly prime bi-ideal of $M$ and let $B_{1}$ and $B_{2}$ be any two bi-ideals of $M$. Then $\overline{\theta}(B_{1})$ and $\overline{\theta}(B_{2})$ are also bi-ideals of $M$ such that $\overline{\theta}(B_{1}) \cap \overline{\theta}(B_{2}) \subseteq \overline{\theta}(B)$. Since $M$ is regular and intra-regular, by Theorem 3.21 $(\overline{\theta}(B_{1})\Gamma \overline{\theta}(B_{2})) \cap (\overline{\theta}(B_{1}) \Gamma \overline{\theta}(B_{2})) = \overline{\theta}(B_{1}) \cap \overline{\theta}(B_{2}) \subseteq \overline{\theta}(B)$. Thus by hypothesis either $\overline{\theta}(B_{1}) \subseteq \overline{\theta}(B)$
or $\overline{\vartheta}(B) \subseteq \overline{\theta}(B)$. Hence $\overline{\theta}(B)$ is strongly irreducible. Similarly $\theta(B)$ is strongly irreducible. Therefore $\vartheta(B)$ is strongly rough irreducible.

References


Compatible Mappings and Common Fixed Points in Fuzzy Metric Spaces

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Abstract:
In this paper, we state and prove some common fixed point Theorems in fuzzy metric spaces in the sense of Kramosil and Michalek, using the notions of compatibility and reciprocal continuity of maps. Our work contains proper generalizations of many important Theorems mainly due to Murthy, Jungck and Cho. We deduce some Corollaries to our Theorems and also illustrate them with suitable examples.

Keywords:
Fuzzy metric spaces, compatible maps, reciprocal continuous maps, common fixed points.

1. Introduction

The concept of fuzzy metric spaces was introduced in various ways by several authors such as Deng [5], Erceg [6], Kaleva and Seikkala [17], Kramosil and Michalek [18]. In particular, Kramosil and Michalek introduced the notion of fuzzy metric spaces in the year 1975 by generalizing the concept of probabilistic metric spaces introduced by Menger, to fuzzy setting. George and Veeramani [7] modified the notion of fuzzy metric spaces introduced by Kramosil and Michalek with the help of continuous $t$-norm and obtained a Hausdorff topology for this kind of fuzzy metric spaces and proved that every metric $d$ on $X$ induces a fuzzy metric $M_d$ on $X$. Many authors think that the George and Veeramani’s definition is an appropriate notion of metric fuzziness in the sense that it provides rich topological structures which can be obtained mainly from the classical theorems.
In recent years, the study of common fixed point theorems satisfying some contractive-type condition has been at the center stage of some intense research activity and a significant number of interesting results have been obtained by various authors such as [1, 2, 11, 22, 29], etc. Jungck [13] introduced the notion of compatible maps by generalizing the concept of commuting maps and established some important common fixed point theorems in a series of his paper [12, 13, 14, 15] satisfying various compatibility conditions. Recently, Singh et al. [27, 28] extended the concept of compatibility and semi-compatibility of maps to fuzzy metric spaces and proved some common fixed point theorems.

Vasuki [30] has generalized the common fixed point theorem of Pant [22] to fuzzy metric spaces. Cho and Jung [2] established some common fixed point theorems for four weakly compatible maps of an \( \varepsilon \)-chainable fuzzy metric space. Mishra et al. [20] used reciprocal continuity while proving some common fixed point theorems in fuzzy metric spaces and showed that the notion of reciprocal continuity can widen the scope of many interesting fixed point theorems in fuzzy metric spaces. Recent literatures on fixed point theory in fuzzy metric spaces can also be viewed in [3, 4, 9, 10, 19]. For basic analysis, we refer to [24].

Our present work is in the direction of extending the classical common fixed point theorems mainly due to Murthy [21] and Jungck [15] to fuzzy metric spaces in the sense of Kramosil and Michalek. The structure of the paper is as follows. After the preliminaries, we prove the fuzzy analogues of Murthy and Jungck’s common fixed point Theorem. We use a weaker notion of ‘reciprocal continuity’ in place of ‘continuity’ in our Theorems to obtain proper generalizations. We have deduced some Corollaries including the fuzzy Banach contraction Theorem due to Grabiec [8] and have incorporated some examples to support our results.

2. Preliminaries

In this section, we recall some definitions and important results which are already in the literature.

**Definition 2.1** [31]. A fuzzy set \( A \) in \( X \) is a mapping \( A : X \rightarrow [0,1] \). For \( x \in X \), \( A(x) \) is called the grade of membership of \( x \).

**Definition 2.2** [26]. A binary operation \( \ast : [0,1] \times [0,1] \rightarrow [0,1] \) is called a continuous \( t \)-norm, if \( ([0,1],\cdot) \) is an abelian topological monoid with unity \( 1 \) such that \( a \ast b \leq c \ast d \), whenever \( a \leq c, b \leq d \), for all \( a, b, c, d \in [0,1] \).

**Example 2.3.** \( a \ast b = \min \{a,b\}, a \ast b = ab, a, b \in [0,1] \) are continuous \( t \)-norms.

**Definition 2.4** [18]. The 3-tuple \((X, M, \ast)\) is called a fuzzy metric space, if \( X \) is an arbitrary set, \( \ast \) is a continuous \( t \)-norm and \( M \) is a fuzzy set in \( X^2 \times [0,\infty[ \), satisfying the following conditions:
- For all \( x, y, z \in X \) and \( s, t > 0 \),
\( M(x, y, 0) = 0 \) \hspace{1cm} (2.4.1)

\( M(x, y, t) = 1, \) for all \( t > 0, \) if and only if \( x = y, \) \hspace{1cm} (2.4.2)

\( M(x, y, t) = M(y, x, t), \) \hspace{1cm} (2.4.3)

\( M(x, z, s + t) \geq M(x, y, s) \ast M(y, z, t) \) \hspace{1cm} (2.4.4)

\( M(x, y) : [0, \infty] \to [0, 1] \) is left continuous, and

\( \lim_{t \to 0^+} M(x, y, t) = 1 \) \hspace{1cm} (2.4.5)

**Example 2.5** [7]. Let \((X, d)\) be a metric space and \(a \ast b = \min\{a, b\} \) or \( ab \) for every \( a, b \in [0, 1]. \) Let \( M_d \) be a fuzzy set in \( X^2 \times [0, \infty[ \) given by \( M_d(x, y, t) = \frac{t}{t + d(x, y)} \) if \( t > 0 \) and \( M_d(x, y, 0) = 0. \) Then \((X, M_d, \ast)\) is a fuzzy metric space and \( M_d \) is called the standard fuzzy metric induced by \( d. \) Thus every metric \( d \) induces a fuzzy metric \( M_d \) on \( X. \) For further examples and elementary properties, we refer to [25].

**Definition 2.6** [8]. A sequence \( \{x_n\} \) in a fuzzy metric space \((X, M, \ast)\) is called

(a) a **Cauchy Sequence**, if \( \lim_{n \to \infty} M(x_n, x_{n+p}, t) = 1, \) for all \( t > 0, \) \( p > 0. \)

(b) convergent to \( x \) (in symbols, \( \lim_{n \to \infty} x_n = x \) or \( x_n \to x \)), if \( \lim_{n \to \infty} M(x_n, x, t) = 1, \) for all \( t > 0. \)

**Definition 2.7** [8]. A fuzzy metric space \((X, M, \ast)\) is said to be **complete**, if every Cauchy sequence in \( X \) is convergent.

**Definition 2.8.** A self map \( T \) of a fuzzy metric space \((X, M, \ast)\) is said to be **continuous**, if for every \( x \in X, \) \( x_n \to x \) implies \( Tx_n \to Tx. \)

**Definition 2.9.** A pair \([A, B]\) of self maps of a fuzzy metric space \((X, M, \ast)\) is said to be

(a) compatible [28], if \( \lim_{n \to \infty} M(ABx_n, BAx_n, t) = 1, \) for all \( t > 0, \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = x, \) for some \( x \in X. \)

(b) semi-compatible [27], if \( \lim_{n \to \infty} ABx_n = Bx, \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = x, \) for some \( x \in X. \)

(c) weak compatible or coincidentally commuting [16], if they commute at their points of coincidence, i.e., for every \( x \in X, \) \( Ax = Bx \) implies \( ABx = BAx. \)

(d) reciprocal continuous [23], if \( \lim_{n \to \infty} ABx_n = Ax, \) \( \lim_{n \to \infty} BAx_n = Bx, \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = x, \) for some \( x \in X. \)
Lemma 2.10. If $[A, B]$ is a pair of compatible self maps of a fuzzy metric space $(X, M, *)$, then they are weak compatible. But the converse is not true. For example [2], let $X = [0, 2]$ and $M$ be the standard fuzzy metric on $X$ induced by the metric $d(x, y) = |x - y|$. Let $A, B$ be self maps of $X$ given by

$$A_x = \begin{cases} 
2 - x, & \text{if } 0 \leq x \leq 1 \\
2, & \text{if } 1 < x \leq 2 
\end{cases} \quad \text{and} \quad B_x = \begin{cases} 
x, & \text{if } 0 \leq x \leq 1 \\
2, & \text{if } 1 < x \leq 2 
\end{cases}$$

For $x_n = 1 - 1/n \in X$, we get, $Ax_n \to 1$, $Bx_n \to 1$, but $\lim_{n \to \infty} M(ABx_n, BAx_n, t) \neq 1$. Therefore $[A, B]$ is not compatible. It is easy to see that $[A, B]$ is weak compatible.

Lemma 2.11. Let $(X, M, *)$ be a fuzzy metric space. If there exists $0 < k < 1$ such that $M(x, y, kt) \geq M(x, y, t)$, for all $t > 0$, then $x = y$.

Lemma 2.12. If $\{y_n\}$ is a sequence in a fuzzy metric space $(X, M, *)$ such that there exists $0 < k < 1$ satisfying $M(y_n, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$ for all $t > 0$, $n > 0$, then $\{y_n\}$ is a Cauchy sequence.

Lemma 2.13 [1]. Let $[A, B]$ be a compatible pair of self maps of a fuzzy metric space $(X, M, *)$ such that $A$ is continuous. If $Ax_n, Bx_n \to z$, for some $z \in X$, then $BAx_n \to Az$.

Lemma 2.14. If $[A, B]$ is a pair of continuous self maps of a fuzzy metric space $(X, M, *)$, then $[A, B]$ is reciprocal continuous. But the converse is not true. For example [20], let $X = [2, 20]$ and $M$ be the standard fuzzy metric on $X$ induced by the metric $d(x, y) = |x - y|$. Let $[A, B]$ be a pair of self maps of $X$ given by

$$A_x = \begin{cases} 
2, & \text{if } x = 2 \\
3, & \text{if } x > 2 
\end{cases} \quad \text{and} \quad B_x = \begin{cases} 
2, & \text{if } x = 2 \\
6, & \text{if } x > 2 
\end{cases}$$

Then $[A, B]$ is reciprocal continuous but not continuous.

Lemma 2.15 [20]. Let $[A, B]$ be a pair of reciprocal continuous self maps of a fuzzy metric space $(X, M, *)$. Then $[A, B]$ is semi-compatible if and only if $[A, B]$ is compatible.
Theorem 2.16 (Murthy [21]). Let \((X,d)\) be a complete metric space and \(f, g\) be self mappings of \(X\) such that the ordered pair \((f, g)\) is weak compatible and satisfy the condition:

For \(x, y \in X\),
\[
d(fx, fy) \leq \lambda \cdot \varphi\left(\max\left\{d(gx, gy), d(fx, gy), d(fx, gx)\right\}\right)
\]
where \(\lambda \in [0, 1)\) and \(\varphi : \mathbb{R}_+ \to \mathbb{R}_+\) is a continuous mapping such that \(\varphi(t) < t\) for every \(t > 0\). If the range of \(g\) contains the range of \(f\) and if \(f\) is continuous, then \(f\) and \(g\) have a unique common fixed point.

Theorem 2.17 (Jungck [15]). Let \(A, B, S \) and \(T\) be self maps of a complete metric space \((X,d)\). Suppose that \(S\) and \(T\) are continuous, the pairs \([A,S]\) and \([B,T]\) are compatible and \(AX \subseteq TX\), \(BX \subseteq SX\). If there exists \(r \in (0,1)\) such that
\[
d(Ax, By) \leq r \max M_{xy}, \quad \text{for } x, y \in X,
\]
where \(M_{xy} = \left\{d(Ax, Sx), d(By, Ty), d(Sx, Ty), \frac{1}{2} \left(d(Ax, Ty) + d(By, Sx)\right)\right\}\), then there is a unique point \(z\) in \(X\) such that \(z = Az = Bz = Sz = Tz\).

In what follows, we assume that \((X,M,*)\) is a fuzzy metric space with the condition:
\[
x_n \to x, \quad y_n \to y \text{ in } X \implies M(x_n, y_n, t) \to M(x, y, t) \quad (2.17.1)
\]

3. Main results

In this section, we state and prove the main results of our paper. We extend the Theorems (2.16) and (2.17) to fuzzy metric spaces and deduce some Corollaries. In particular, we deduce the fuzzy Banach contraction Theorem due to Grabiec [8] as a Corollary. We further construct examples to illustrate our results.

Theorem 3.1. Let \(A, B\) be self maps of a complete fuzzy metric space \((X,M,*)\) satisfying the following conditions:

\[
AX \subseteq BX, \quad (3.1.1)
\]

\([A,B]\) is compatible, \quad (3.1.2)

\(A\) or \(B\) is continuous, and \quad (3.1.3)

\[
M(Ax, Ay, kt) \geq \min\{M(Bx, By, t), M(Ax, By, t), M(Ax, Bx, t)\}, \quad (3.1.4)
\]

for all \(x, y \in X\), \(t > 0\) and some \(0 < k < 1\).

Then \(A\) and \(B\) have a unique common fixed point.

Proof. Let \(x_0 \in X\) be arbitrary. Then \(Ax_0 = Bx_1\) for some \(x_1 \in X\) by (3.1.1) and so on. We thus obtain a sequence \(\{y_n\}\) in \(X\) such that \(y_n = Ax_{n+1} = Bx_n\), \(n \in \mathbb{N}\). We get
\[
M(y_{n+1}, y_n, kt) = M(Ax_{n+1}, Ax_n, kt) \geq \min\{M(Bx_{n+1}, Bx_n, t), M(Ax_{n+1}, Bx_n, t)\},
\]
\(M(Ax_{n-1}, Bx_{n-1}, t)\) by (3.1.4). Thus \(\min\{M(y_{n-1}, y_n, t), M(y_n, y_{n-1}, t)\}\) = \(M(y_{n-1}, y_n, t)\), for every \(t > 0\). Therefore, by Lemma (2.12), \(\{y_n\}\) is a Cauchy sequence in \(X\). As \(X\) is complete, \(y_n \to y\) for some \(y \in X\), i.e., \(y_n = Ax_{n-1} = Bx_n \to y\).

(a) Let \(A\) be continuous. As \([A, B]\) is compatible, we have by Lemma (2.13), \(BAx_n \to Ay\). Now \(M(Ax_n, AAx_n, kt) \geq \min\{M(Bx_n, BAX_n, t), M(Ax_n, BAX_n, t), M(Ax_n, Bx_n, t)\}\), for every \(t > 0\). Letting \(n \to \infty\), we get, \(M(y, Ay, kt) \geq \min\{M(y, Ay, t), M(y, Ay, t)\}\) = \(M(y, Ay, t)\), for every \(t > 0\). Therefore, by Lemma (2.11), we get \(Ay = y\). By (3.1.1), we get \(y = Ay = By\) for some \(y \in X\). Also, \(M(AAx_n, Ay, t) \geq \min\{M(BAX_n, By, t/k), M(AAx_n, By, t/k), M(AAx_n, Bx_n, t/k)\}\). On letting \(n \to \infty\), we get \(M(Ay, Ay, t) \geq \min\{M(Ay, By, t/k), M(Ay, By, t/k), M(Ay, Ay, t/k)\}\) = 1. Therefore, \(Ay = Ay\) and thus \(y = Ay = Ay = By\). Now we get, \(M(Ay, By, t) = M(Ay, Ay, t)\), by (3.1.2) and Lemma (2.10).

(b) Let \(B\) be continuous. As \([A, B]\) is compatible, we have Lemma (2.13), \(ABx_n \to By\). Now \(M(Ax_n, AAX_n, kt) \geq \min\{M(Bx_n, BAX_n, t), M(Ax_n, BAX_n, t), M(Ax_n, Bx_n, t)\}\), which in the limit as \(n \to \infty\), gives \(M(y, By, kt) \geq \min\{M(y, By, t), M(y, By, t)\}\), for every \(t > 0\). Therefore, by Lemma (2.11), we get, \(By = y\). Now we get, \(M(ABx_n, Ay, t) \geq \min\{M(BAX_n, By, t/k), M(ABx_n, By, t/k), M(ABx_n, Bx_n, t/k)\}\). Letting \(n \to \infty\), we get \(M(By, Ay, t) \geq \min\{M(By, By, t/k), M(By, By, t/k), M(By, By, t/k)\}\) = 1, for every \(t > 0\). Therefore, \(By = Ay\) and thus \(y = Ay = By\), a common fixed point of \(A\) and \(B\).

(c) Uniqueness. Let \(w \in X\) be such that \(Aw = Bw = w\). Now, \(M(y, w, kt) \geq \min\{M(By, Bw, t), M(Ay, Bw, t), M(Ay, By, t)\}\) = \(\min\{M(y, w, t), M(y, w, t)\}\), for every \(t > 0\). Therefore, \(y = w\) and so the common fixed point is unique.

Considering \(B = I\), the identity map of \(X\), we get the following Corollary.

**Corollary 3.2.** Let \(A\) be a self map of a complete fuzzy metric space \((X, M, *)\) such that \(M(Ax, Ay, kt) \geq \min\{M(x, y, t), M(Ax, y, t), M(Ax, x, t)\}\), for all \(x, y \in X, t > 0\) and some \(0 < k < 1\). Then \(A\) has a unique fixed point.

The following result is the fuzzy Banach contraction Theorem due to Grabiec [8].
Corollary 3.3. Let $A$ be a self map of a complete fuzzy metric space $(X, M, *)$ such that $M(Ax, Ay, kt) \geq M(x, y, t)$ for all $x, y \in X$, $t > 0$ and some $0 < k < 1$. Then $A$ has a unique fixed point.

Proof. It follows immediately from the Corollary 3.2.

Example 3.4. Let $X = [0, 1]$ and $M$ be the standard fuzzy metric on $X$ induced by $d(x, y) = |x - y|$. Now $(X, M, *)$ is a complete fuzzy metric space. Let $A, B$ be self maps of $X$ given by $Ax = 1$ if $x$ is rational and $Bx = 0$, if $x$ is irrational. Then $A, B$ satisfy all the conditions of the Theorem (3.1) and $1 \in X$ is the unique common fixed point $A$ and $B$.

Theorem 3.5. Let $A, B, S$ and $T$ be self maps of a complete fuzzy metric space $(X, M, *)$ satisfying the following conditions:

$$AX \subseteq TX, \ BX \subseteq SX, \quad (3.5.1)$$

The pairs $[A, S]$ and $[B, T]$ are semi-compatible, \quad (3.5.2)

The pairs $[A, S]$ and $[B, T]$ are reciprocal continuous, \quad (3.5.3)

$$M(Ax, By, kt) \geq \min\{M(Ax, Sx, t), M(By, Ty, t), M(Sx, Ty, t)\}, \quad (3.5.4)$$

for every $x, y \in X$, $t > 0$ and some $0 < k < 1$.

Then $A, B, S$ and $T$ have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Then there exists $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$, $Bx_1 = Sx_2$ by (3.5.1), and so on. We thus obtain a sequence $\{y_n\}$ in $X$ such that $y_{2n+1} = Ax_{2n+2} = Tx_{2n+1}$, $y_{2n} = Bx_{2n+1} = Sx_{2n}$, $n \in \mathbb{N}$. We get $M(y_{2n}, y_{2n+1}, kt) = M(Bx_{2n+1}, Ax_{2n+2}, kt) \geq \min\{M(Ax_{2n}, Sx_{2n}, t), M(Bx_{2n+1}, Tx_{2n+1}, t), M(Sx_{2n}, Tx_{2n+1}, t)\} = \min\{M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t)\}$, for every $t > 0$. Therefore, we get $M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n+1}, t)$, for every $t > 0$, $n > 0$. Similarly, $M(y_{2n-1}, y_{2n}, kt) \geq M(y_{2n-2}, y_{2n-1}, t)$, for every $t > 0$, $n > 0$. These give, $M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_{n}, t)$, for every $t > 0$, $n > 0$. Therefore, by Lemma (2.12), $\{y_n\}$ is a Cauchy sequence in $X$. As $X$ is complete, $\{y_n\}$ finds a limit $z$ in $X$. Consequently, all the subsequences $\{Ax_{2n}\}, \{Sx_{2n}\}, \{Bx_{2n+1}\}, \{Tx_{2n+1}\}$ converge to $z$. As $[A, S]$ is reciprocal continuous and compatible (by Lemma (2.15) and (3.5.2)), we get $ASx_{2n} \rightarrow Az$, $Sx_{2n} \rightarrow Sz$, $M(ASx_{2n}, Sx_{2n}, t) \rightarrow 1$, for every $t > 0$. Therefore, by (2.17.1), we get $M(Az, Sz, t) = 1$, for every $t > 0$ and so $Az = Sz$. By a similar argument, we get $Bz = Tz$.\]
Now, \( M(Ax_{2n}, Bz, kt) \geq \min \left\{ M(Ax_{2n}, Sz_{2n}, t), M(Bz, Tz, t), M(Sz_{2n}, Tz, t) \right\} \), by (3.5.4). Letting \( n \to \infty \), we get \( M(z, Bz, kt) \geq \min \left\{ M(z, z, t), M(Bz, Tz, t), M(z, Tz, t) \right\} = M(z, Tz, t) = M(z, Bz, t) \), for every \( t > 0 \). Therefore, we get \( z = Bz \) and thus \( z = Bz = Tz \). Again, \( M(Az, z, kt) = M(Az, Bz, kt) \geq \min \left\{ M(Az, Sz, t), M(Bz, Tz, t), M(Sz, Tz, t) \right\} = M(Sz, Tz, t) = M(Az, z, t) \). Therefore, \( Az = z \) and so \( Az = Bz = Sz = Tz = z \), a common fixed point of \( A, B, S \) and \( T \). To show uniqueness, let \( w \in X \) such that \( Aw = Bw = Sw = Tw = w \). Now, \( M(z, w, kt) = M(Az, Bw, kt) \geq \min \left\{ M(Az, Sz, t), M(Bw, Tw, t), M(Sz, Tw, t) \right\} = \min \left\{ M(z, z, t), M(w, w, t), M(z, w, t) \right\} = M(z, w, t) \), for every \( t > 0 \). Therefore \( z = w \) and so the common fixed point is unique.

**Corollary 3.6.** Let \( A, B \) be two continuous self maps of a complete fuzzy metric space \((X, M, \ast)\) such that \( M(Ax, By, kt) \geq \min \left\{ M(Ax, x, t), M(By, y, t), M(x, y, t) \right\} \), for every \( x, y \in X \), \( t > 0 \) and some \( 0 < k < 1 \). Then \( A \) and \( B \) have a unique common fixed point.

**Proof.** Taking \( S = T = I \), the identity map of \( X \), the result immediately follows from the Theorem 3.5.

With \( A = B \), we get the following Corollary.

**Corollary 3.7.** Let \( A \) be a continuous self map of a complete fuzzy metric space \((X, M, \ast)\) satisfying \( M(Ax, Ay, kt) \geq \min \left\{ M(Ax, x, t), M(Ay, y, t), M(x, y, t) \right\} \), for all \( x, y \in X \), \( t > 0 \) and some \( 0 < k < 1 \). Then \( A \) has a unique fixed point.

We now deduce the fuzzy Banach contraction Theorem due to Grabiec [8] as a Corollary.

**Corollary 3.8.** Let \( A \) be a self map of a complete fuzzy metric space \((X, M, \ast)\) such that \( M(Ax, Ay, kt) \geq M(x, y, t) \), for all \( x, y \in X \), \( t > 0 \) and some \( 0 < k < 1 \). Then \( A \) has a unique fixed point.

**Proof.** It follows immediately from Corollary 3.7 as such a mapping is obviously continuous.

**Example 3.9.** Let \( X = [0,1] \) and \( M \) be the standard fuzzy metric on \( X \) induced by \( d(x, y) = |x - y| \). Now \((X, M, \ast)\) is a complete fuzzy metric space. Let \( A, B, S, T \) be self maps of \( X \) given by \( Ax = x/81 \), \( Bx = x/27 \), \( Sx = x/9 \), \( Tx = x/3 \). Then \( A, B, S, T \) satisfy all the conditions of the Theorem 3.5 and have a unique common fixed point \( 0 \in X \). It may be verified that \( M(Ax, By, kt) \geq M(Sx, Ty, t) \), for all \( x, y \in X \), \( t > 0 \) with \( k = 1/3 \). So the condition (3.5.4) is satisfied.
**Remarks.** The Theorem 3.1 and 3.5 remain true if the condition \( (X, M, *) \) is a complete fuzzy metric space is replaced by “the range of one of the maps is a complete subspace of \( X \)”. Also the Theorem 3.1 cannot be deduced as a particular case of the Theorem 3.5. Moreover, the \( t \)-norm \(*\) we have used is an arbitrary continuous \( t \)-norm.

**References**


Bipolar Fuzzy Structure of $BG$-subalgebras

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Abstract:
Based on the concept of bipolar fuzzy set, a theoretical approach of the $BG$-subalgebras is established. Some characterizations of bipolar fuzzy $BG$-subalgebras of $BG$-algebras are given.

Key words and phrases:
$BG$-algebra, $BG$-subalgebra, bipolar fuzzy $BG$-subalgebra.

1. Introduction

Since the fuzzy set was introduced by Zadeh [22], many new approaches and theories treating imprecision and uncertainty have been proposed, such as the generalized theory of uncertainty (GTU) introduced by Zadeh [23] and the intuitionistic fuzzy sets introduced by Atanassov [2] and so on. Among these theories, a well-known extension of the classic fuzzy set is bipolar fuzzy set theory, which was pioneered by Zhang [25]. Since then, many researchers have investigated this topic and obtained some meaningful conclusions [3,4,24,26].

In traditional fuzzy sets the membership degree range is $[0,1]$. The membership degree is the degree of belongingness of an element to a set. The membership degree 1 indicates that an element completely belongs to its corresponding set, the membership degree 0 indicates that an element does not belong to the corresponding set and the membership degree on the interval $(0,1)$ indicates the partial membership to the corresponding set. Sometimes, membership degree also means the satisfaction degree of elements to some property corresponding to a set and its counter property. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is increased from the interval $[0,1]$ to the interval $[-1,1]$. In a bipolar fuzzy set the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on $[0,1]$ indicate that elements somewhat satisfy the property
and the membership degrees on \([-1,0]\) indicate that elements somewhat satisfy the implicit counter-property. Although bipolar fuzzy sets and intuitionistic fuzzy sets look similar to each other, they are essentially different sets \([10]\). In many domains, it is important to be able to deal with bipolar information. It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible.

\(BCK\)-algebras and \(BCI\)-algebras \([5,14]\) are two important classes of logical algebras introduced by Imai and Iseki. It is known that the class of \(BCK\)-algebra is a proper subclass of the class of \(BCI\)-algebras. Neggers and Kim \([12]\) introduced a new notion, called a \(B\)-algebra which is related to several classes of algebras of interest such as \(BCI/BCK\)-algebras. Kim and Kim \([7]\) introduced the notion of \(BC\)-algebras, which is a generalization of \(B\)-algebras. Ahn and Lee \([1]\) fuzzified \(BG\)-algebras. Saed \([13]\) introduced fuzzy topological \(BG\)-algebras. Senapati et al. \([17, 20, 21]\) done lot of works on \(B\)-algebras. They also \([15, 16, 18, 19]\) presented the concept and basic properties of intuitionistic fuzzy subalgebras, intuitionistic \(L\)-fuzzy ideals, interval-valued intuitionistic fuzzy subalgebras, interval-valued intuitionistic fuzzy closed ideals of \(BG\)-algebras. Lee \([8]\) introduced the notion of bipolar fuzzy subalgebras and ideals in \(BCK/BCI\)-algebras. The concept of bipolar valued fuzzy translation and bipolar valued fuzzy \(S\)-extension of a bipolar valued fuzzy subalgebra in \(BCK/BCI\)-algebra was introduced by Jun et al. \([6]\). Motivated by this, in this paper, the notions of bipolar fuzzy \(BG\)-subalgebras is introduced and their properties are investigated.

2. Preliminaries

In this section, some elementary aspects that are necessary for this paper are included. A \(BG\)-algebra is an important class of logical algebras introduced by Kim and Kim \([7]\) and was extensively investigated by several researchers. This algebra is defined as follows.

A non-empty set \(X\) with a constant 0 and a binary operation \(\ast\) is called to be \(BG\)-algebra \([7]\) if it satisfies the following axioms

\[F1.\quad x \ast x = 0\]

\[F2.\quad x \ast 0 = x\]

\[F3.\quad (x \ast y) \ast (0 \ast y) = x, \quad \text{for all } x, y \in X.\]

In any \(BG\)-algebra \(X\), the following hold (see \([7]\)):

(i) the right cancellation law holds in \(X\), i.e., \(x \ast y = z \ast y\) implies \(x = z\),

(ii) 0 \((0 \ast x) = x\) for all \(x \in X\),

(iii) if \(x \ast y = 0\), then \(x = y\) for any \(x, y \in X\),

(iv) if \(0 \ast x = 0 \ast y\), then \(x = y\) for any \(x, y \in X\),

(v) \((x \ast (0 \ast x)) \ast x = x\) for all \(x \in X\).

A non-empty subset \(S\) of a \(BG\)-algebra \(X\) is called a subalgebra \([1]\) of \(X\) if \(x \ast y \in S\) for any \(x, y \in S\).
**Definition 2.1** [22] (Fuzzy set). Let \( X \) be the collection of objects denoted generally by \( x \), then a fuzzy set \( A \) in \( X \) is defined as \( A = \{ (x, \mu_A(x)) : x \in X \} \), where \( \mu_A(x) \) is called the membership value of \( x \) in \( A \) and \( 0 \leq \mu_A(x) \leq 1 \).

Combined the definitions of \( BG \)-subalgebra over crisp set and the idea of fuzzy set Ahn and Lee [1] defined fuzzy \( BG \)-subalgebra, which is defined below.

**Definition 2.2** [1] (Fuzzy \( BG \)-subalgebra). Let \( \mu \) be a fuzzy set in a \( BG \)-algebra. Then \( \mu \) is called a fuzzy \( BG \)-subalgebra of \( X \) if \( \mu(x \circ y) \geq \min\{\mu(x), \mu(y)\} \) for all \( x, y \in X \), where \( \mu(x) \) is the membership value of \( x \) in \( X \).

**Definition 2.3** [25] (Bipolar fuzzy set). Let \( X \) be a nonempty set. A bipolar fuzzy set \( \varphi \) in \( X \) is an object having the form \( \varphi = \{ (x, \varphi^+(x), \varphi^-(x)) : x \in X \} \) where \( \varphi^+: X \rightarrow [0,1] \) and \( \varphi^-: X \rightarrow [0,1] \) are mappings.

We use the positive membership degree \( \varphi^+(x) \) to denote the satisfaction degree of an element \( x \) to the property corresponding to a bipolar fuzzy set \( \varphi \) and the negative membership degree \( \varphi^-(x) \) to denote the satisfaction degree of an element \( x \) to some implicit counter-property corresponding to a bipolar fuzzy set \( \varphi \). If \( \varphi^+(x) \neq 0 \) and \( \varphi^-(x) = 0 \), it is the situation that \( x \) is regarded as having only positive satisfaction for \( \varphi \). If \( \varphi^+(x) = 0 \) and \( \varphi^-(x) \neq 0 \), it is the situation that \( x \) does not satisfy the property of \( \varphi \) but somewhat satisfies the counter property of \( \varphi \). It is possible for an element \( x \) to be such that \( \varphi^+(x) \neq 0 \) and \( \varphi^-(x) \neq 0 \) when the membership function of the property overlaps that of its counter property over some portion of \( X \).

**Definition 2.4** [9]. Let \( \varphi_1 = (\varphi_1^+, \varphi_1^-) \) and \( \varphi_2 = (\varphi_2^+, \varphi_2^-) \) be two bipolar fuzzy sets on \( X \). Then the intersection and union of \( \varphi_1 \) and \( \varphi_2 \) is denoted by \( \varphi_1 \cap \varphi_2 \) and \( \varphi_1 \cup \varphi_2 \) respectively and is given by

\[
\varphi_1 \cap \varphi_2 = (\varphi_1^+ \cap \varphi_2^+, \varphi_1^- \cap \varphi_2^-) = \left\{ \min\{\varphi_1^+ \cap \varphi_2^+, \varphi_1^- \cap \varphi_2^-\} \right\}
\]

\[
\varphi_1 \cup \varphi_2 = (\varphi_1^+ \cup \varphi_2^+, \varphi_1^- \cup \varphi_2^-) = \left\{ \max\{\varphi_1^+ \cup \varphi_2^+, \varphi_1^- \cup \varphi_2^-\} \right\}.
\]

3. Main results

For the sake of simplicity, we shall use the symbol \( \varphi = (\varphi^+, \varphi^-) \) for the bipolar fuzzy subset \( \varphi = \{ (x, \varphi^+(x), \varphi^-(x)) : x \in X \} \). Throughout this paper, \( X \) always means a \( BG \)-algebra without any specification.
Definition 3.1. Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy set in $X$, then the set $\varphi$ is bipolar fuzzy $BG$-subalgebra over the binary operator $*$ if it satisfies the following conditions:

(IBS1) $\varphi^+ (x \ast y) \geq \min \{ \varphi^+ (x), \varphi^+ (y) \}$

(IBS2) $\varphi^- (x \ast y) \leq \max \{ \varphi^- (x), \varphi^- (y) \}$

for all $x, y \in X$.

We consider an example of bipolar fuzzy $BG$-subalgebra below.

Example 3.2. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a $BG$-algebra with the following Cayley table:

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Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy set in $X$ defined by $\varphi^+ (x) = \begin{cases} 0.55, & \text{if } x \in \{0, 2, 4\} \\ 0.46, & \text{otherwise} \end{cases}$ and $\varphi^- (x) = \begin{cases} -0.42, & \text{if } x \in \{0, 2, 4\} \\ -0.36, & \text{otherwise} \end{cases}$.

All the conditions of Definition 3.1 have been satisfied by the set $\varphi$. Thus $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $BG$-subalgebra of $X$.

Proposition 3.3. If $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $BG$-subalgebra in $X$, then for all $x \in X$, $\varphi^+ (0) \geq \varphi^+ (x)$ and $\varphi^- (0) \leq \varphi^- (x)$. Thus, $\varphi^+ (0)$ and $\varphi^- (0)$ are the upper bounds and lower bounds of $\varphi^+ (x)$ and $\varphi^- (x)$ respectively.

Proof. Let $x \in X$. Then $\varphi^+ (0) = \varphi^+ (x \ast x) \geq \min \{ \varphi^+ (x), \varphi^+ (x) \} = \varphi^+ (x)$ and $\varphi^- (0) = \varphi^- (x \ast x) \leq \max \{ \varphi^- (x), \varphi^- (x) \} = \varphi^- (x)$.

Theorem 3.4. Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy $BG$-subalgebra of $X$. If there exists a sequence $x_n$ in $X$ such that $\lim_{n \to \infty} \varphi^+ (x_n) = 1$ and $\lim_{n \to \infty} \varphi^- (x_n) = -1$, then $\varphi^+ (0) = 1$ and $\varphi^- (0) = -1$. 
Proof. By Proposition 3.3, \( \varphi^+ (0) \geq \varphi^+ (x) \) for all \( x \in X \), therefore \( \varphi^+ (0) \geq \lim_{n \to \infty} \varphi^+ (x_n) = 1 \). Hence, \( \varphi^+ (0) = 1 \).

Again, by Proposition 3.3, \( \varphi^- (0) \leq \varphi^- (x) \) for all \( x \in X \), thus \( \varphi^- (0) \leq \varphi^- (x_n) \) for every positive integer \( n \). Now, \( -1 \leq \varphi^- (0) \leq \lim_{n \to \infty} \varphi^- (x_n) = -1 \). Hence, \( \varphi^- (0) = -1 \).

**Proposition 3.5.** If a bipolar fuzzy set \( \varphi = (\varphi^+, \varphi^-) \) in \( X \) is a bipolar fuzzy BG-subalgebra, then \( \varphi^+ (0 \ast x) \geq \varphi^+ (x) \) and \( \varphi^- (0 \ast x) \leq \varphi^- (x) \) for all \( x \in X \).

**Proof.** For all \( x \in X \), \( \varphi^+ (0 \ast x) \geq \min \{ \varphi^+ (0), \varphi^+ (x) \} = \min \{ \varphi^+ (x \ast x), \varphi^+ (x) \} \geq \min \{ \varphi^+ (x), \varphi^+ (x) \} = \varphi^+ (x) \) and \( \varphi^- (0 \ast x) \leq \max \{ \varphi^- (0), \varphi^- (x) \} = \max \{ \varphi^- (x \ast x), \varphi^- (x) \} = \varphi^- (x) \).

For any elements \( x \) and \( y \) of \( X \), let us write \( \prod^n x \ast y \) for \( x \ast (\cdots (x \ast y)) \) where \( x \) occurs \( n \) times.

**Theorem 3.6.** Let \( \varphi = (\varphi^+, \varphi^-) \) be a bipolar fuzzy BG-subalgebra of \( X \) and let \( n \in \mathbb{N} \) (the set of natural numbers). Then

(i) \( \varphi^+ (\prod^n x \ast x) \geq \varphi^+ (x) \), for any odd number \( n \),

(ii) \( \varphi^- (\prod^n x \ast x) \leq \varphi^- (x) \), for any odd number \( n \),

(iii) \( \varphi^+ (\prod^n x \ast x) = \varphi^+ (x) \), for any even number \( n \),

(iv) \( \varphi^- (\prod^n x \ast x) = \varphi^- (x) \), for any even number \( n \).

**Proof.** Let \( x \in X \) and assume that \( n \) is odd. Then \( n = 2p - 1 \) for some positive integer \( p \). We prove the theorem by induction. Now \( \varphi^+ (x \ast x) = \varphi^+ (0) \geq \varphi^+ (x) \) and \( \varphi^- (x \ast x) = \varphi^- (0) \leq \varphi^- (x) \). Suppose that \( \varphi^+ (\prod^{2p-1} x \ast x) \geq \varphi^+ (x) \) and \( \varphi^- (\prod^{2p-1} x \ast x) \leq \varphi^- (x) \).

Then by assumption,

\[
\varphi^+ (\prod^{2p-1} x \ast x) = \varphi^+ (\prod^{2p-1} x \ast x) = \varphi^+ (\prod^{2p-1} x \ast (x \ast x)) = \varphi^+ (\prod^{2p-1} x \ast x) \geq \varphi^+ (x)
\]

and

\[
\varphi^- (\prod^{2p-1} x \ast x) = \varphi^- (\prod^{2p-1} x \ast x) = \varphi^- (\prod^{2p-1} x \ast (x \ast x)) = \varphi^- (\prod^{2p-1} x \ast x) \leq \varphi^- (x),
\]

which proves (i) and (ii). Proofs are similar for the cases (iii) and (iv).

The intersection of two bipolar fuzzy BG-subalgebras is also a bipolar fuzzy BG-subalgebra, which is proved in the following theorem.

**Theorem 3.7.** Let \( \varphi_1 = (\varphi^+_1, \varphi^-_1) \) and \( \varphi_2 = (\varphi^+_2, \varphi^-_2) \) be two bipolar fuzzy BG-subalgebras of \( X \). Then \( \varphi_1 \cap \varphi_2 \) is a bipolar fuzzy BG-subalgebra of \( X \).
Proof. Let \( x, y \in \varphi_1 \cap \varphi_2 \). Then \( x, y \in \varphi_1 \) and \( x, y \in \varphi_2 \). Now, \( \varphi_1 \cap \varphi_2 \prec (x \ast y) = \min \{ \varphi_1 \prec (x \ast y) \} \) and \( \varphi_2 \prec (x \ast y) \). Now, \( \varphi_1 \prec (x \ast y) \geq \min \{ \varphi_1 \prec (x), \varphi_1 \prec (y) \} \).

The above theorem can be generalized as follows.

**Theorem 3.8.** Let \( \{ \varphi_i \}_{i=1,2,3,\cdots} \) be a family of bipolar fuzzy \( BG \)-subalgebra of \( X \). Then \( \bigcap \varphi_i \) is also a bipolar fuzzy \( BG \)-subalgebra of \( X \), where \( \bigcap \varphi_i = \left( \min \varphi_i(x), \max \varphi_i(x) \right) \).

The union of any set of bipolar fuzzy \( BG \)-subalgebras need not be a bipolar fuzzy \( BG \)-subalgebra which is shown in the following example.

**Example 3.9.** Let \( X = \{0,1,2,3,4,5\} \) be a \( BG \)-algebra with the following Cayley table:

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Let \( \varphi = (\varphi^+, \varphi^-) \) and \( \psi = (\psi^+, \psi^-) \) be two bipolar fuzzy sets in \( X \) defined by

\[
\varphi^+(x) = \begin{cases} 0.77, & \text{if } x \in \{0,4\} \\ 0.26, & \text{if } x \in X \setminus \{0,4\} \end{cases}, \quad \varphi^-(x) = \begin{cases} -0.52, & \text{if } x \in \{0,3\} \\ -0.13, & \text{if } x \in X \setminus \{0,3\} \end{cases}.
\]

\[
\psi^+(x) = \begin{cases} 0.81, & \text{if } x \in \{0,5\} \\ 0.36, & \text{if } x \in X \setminus \{0,5\} \end{cases}, \quad \psi^-(x) = \begin{cases} -0.44, & \text{if } x \in \{0,4\} \\ -0.18, & \text{if } x \in X \setminus \{0,4\} \end{cases}.
\]

Then \( \varphi = (\varphi^+, \varphi^-) \) and \( \psi = (\psi^+, \psi^-) \) are bipolar fuzzy \( BG \)-subalgebras of \( X \). But the union of \( \varphi \cup \psi \) is not a bipolar fuzzy \( BG \)-subalgebras of \( X \) since \( \varphi^+ \cup \psi^+ \prec (4 \ast 5) = 0.36 \geq 0.81 = \max \{ \varphi^+ \cup \psi^+ (4), \varphi^+ \cup \psi^+ (5) \} \) and \( \varphi^- \cup \psi^- \prec (3 \ast 4) = -0.18 \leq -0.52 = \min \{ \varphi^- \cup \psi^- (3), \varphi^- \cup \psi^- (4) \} \).
The sets \( \{ x \in X : \phi^+ (x) = \phi^+ (0) \} \) and \( \{ x \in X : \phi^- (x) = \phi^- (0) \} \) are denoted by \( I^+ \) and \( I^- \)-respectively. These two sets are also \( BG \)-subalgebra of \( X \).

**Theorem 3.10.** Let \( \varphi = (\varphi^+, \varphi^-) \) be a bipolar fuzzy \( BG \)-subalgebra of \( X \), then the sets \( I^+ \) and \( I^- \) are \( BG \)-subalgebras of \( X \).

**Proof.** Let \( x, y \in I^+ \). Then \( \varphi^+ (x) = \varphi^+ (0) = \varphi^+ (y) \) and so, \( \varphi^+ (x \ast y) \geq \min \{ \varphi^+ (x), \varphi^+ (y) \} = \varphi^+ (0) \). By using Proposition 3.3, we know that \( \varphi^+ (x \ast y) = \varphi^+ (0) \) or equivalently \( x \ast y \in I^+ \).

Again let \( x, y \in I^- \). Then \( \varphi^- (x) = \varphi^- (0) = \varphi^- (y) \) and so, \( \varphi^- (x \ast y) \leq \max \{ \varphi^- (x), \varphi^- (y) \} = \varphi^- (0) \). Then by Proposition 3.3, we know that \( \varphi^- (x \ast y) = \varphi^- (0) \) or equivalently \( x \ast y \in I^- \). Hence, the sets \( I^+ \) and \( I^- \) are \( BG \)-subalgebras of \( X \).

**Theorem 3.11.** Let \( B \) be a nonempty subset of \( X \) and \( \varphi = (\varphi^+, \varphi^-) \) be a bipolar fuzzy set in \( X \) defined by

\[
\varphi^+(x) = \begin{cases} \lambda, & \text{if } x \in B \\ \tau, & \text{otherwise} \end{cases}
\quad \text{and} \quad
\varphi^-(x) = \begin{cases} \gamma, & \text{if } x \in B \\ \delta, & \text{otherwise} \end{cases}
\]

for all \( \lambda, \tau, \gamma, \delta \in [0,1] \) with \( \lambda \geq \tau \) and \( \gamma \leq \delta \). Then \( \varphi \) is a bipolar fuzzy \( BG \)-subalgebra of \( X \) if and only if \( B \) is a \( BG \)-subalgebra of \( X \). Moreover, \( I^+ = B = I^- \).

**Proof.** Let \( \varphi \) be a bipolar fuzzy \( BG \)-subalgebra of \( X \). Let \( x, y \in X \) be such that \( x, y \in B \). Then \( \varphi^+(x \ast y) \geq \min \{ \varphi^+(x), \varphi^+(y) \} = \min \{ \lambda, \lambda \} = \lambda \) and \( \varphi^- (x \ast y) \leq \max \{ \varphi^- (x), \varphi^- (y) \} = \max \{ \gamma, \gamma \} = \gamma \). So \( x \ast y \in B \). Hence, \( B \) is a \( BG \)-subalgebra of \( X \).

Conversely, suppose that \( B \) is a \( BG \)-subalgebra of \( X \). Let \( x, y \in X \). Consider two cases:

**Case (i)** If \( x, y \in B \), then \( x \ast y \in B \), thus \( \varphi^+(x \ast y) = \lambda = \min \{ \varphi^+(x), \varphi^+(y) \} \) and \( \varphi^- (x \ast y) = \gamma = \max \{ \varphi^- (x), \varphi^- (y) \} \).

**Case (ii)** If \( x \notin B \) or, \( y \notin B \), then \( \varphi^+(x \ast y) \geq \tau = \min \{ \varphi^+(x), \varphi^+(y) \} \) and \( \varphi^- (x \ast y) \leq \delta = \max \{ \varphi^- (x), \varphi^- (y) \} \).

Hence, \( \varphi \) is a bipolar fuzzy \( BG \)-subalgebra of \( X \). Also, \( I^+ = \{ x \in X, \varphi^+(x) = \varphi^+(0) \} = \{ x \in X, \varphi^+(x) = \lambda \} = B \) and \( I^- = \{ x \in X, \varphi^- (x) = \varphi^- (0) \} = \{ x \in X, \varphi^- (x) = \gamma \} = B \).
Definition 3.12. Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy $BG$-subalgebra of $X$. For $(s, t) \in [-1, 0] \times [0, 1]$, the set $U(\varphi^+: t) = \{x \in X : \varphi^+(x) \geq t\}$ is called positive $t$-cut of $\varphi$ and $L(\varphi^-: s) = \{x \in X : \varphi^-(x) \leq s\}$ is called negative $s$-cut of $\varphi$.

Theorem 3.13. If $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $BG$-subalgebra of $X$, then the positive $t$-cut and negative $s$-cut of $\varphi$ are $BG$-subalgebras of $X$.

Proof. Let $x, y \in U(\varphi^+: t)$. Then $\varphi^+(x) \geq t$ and $\varphi^+(y) \geq t$. It follows that $\varphi^+(x \ast y) \geq \min \{\varphi^+(x), \varphi^+(y)\} \geq t$ so that $x \ast y \in U(\varphi^+: t)$. Hence, $U(\varphi^+: t)$ is a $BG$-subalgebra of $X$. Let $x, y \in L(\varphi^-: s)$. Then $\varphi^-(x) \leq s$ and $\varphi^-(y) \leq s$. It follows that $\varphi^-(x \ast y) \leq \max \{\varphi^-(x), \varphi^-(y)\} \leq s$ so that $x \ast y \in L(\varphi^-: s)$. Hence, $L(\varphi^-: s)$ is a $BG$-subalgebra of $X$.

Theorem 3.14. Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy set in $X$, such that the sets $U(\varphi^+: t)$ and $L(\varphi^-: s)$ are $BG$-subalgebras of $X$ for every $(s, t) \in [-1, 0] \times [0, 1]$. Then $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $BG$-subalgebra of $X$.

Proof. Let for every $(s, t) \in [-1, 0] \times [0, 1]$, $U(\varphi^+: t)$ and $L(\varphi^-: s)$ are $BG$-subalgebras of $X$. In contrary, let $x_0, y_0 \in X$ be such that $\varphi^+(x_0 \ast y_0) < \min \{\varphi^+(x_0), \varphi^+(y_0)\}$. Let $\varphi^+(x_0) = \theta_1$, $\varphi^+(y_0) = \theta_2$ and $\varphi^+(x_0 \ast y_0) = t$. Then $t < \min \{\theta_1, \theta_2\}$. Let us consider, $t = 1/2[\varphi^+(x_0 \ast y_0) + \min \{\varphi^+(x_0), \varphi^+(y_0)\}]$. We get that $t = 1/2(t + \min \{\theta_1, \theta_2\})$. Therefore, $\theta_1 > t_1 = 1/2(t + \min \{\theta_1, \theta_2\}) > t$ and $\theta_2 > t_2 = 1/2(t + \min \{\theta_1, \theta_2\}) > t$. Hence, $t < \min \{\theta_1, \theta_2\} > t$. Thus $\varphi^+(x \ast y) \leq \min \{\varphi^+(x), \varphi^+(y)\}$ for all $x, y \in X$.

Again let $x_0, y_0 \in X$ be such that $\varphi^-(x_0 \ast y_0) > \max \{\varphi^-(x_0), \varphi^-(y_0)\}$. Let $\varphi^-(x_0) = \eta_1$, $\varphi^-(y_0) = \eta_2$ and $\varphi^-(x_0 \ast y_0) = s$. Then $s > \max \{\eta_1, \eta_2\}$. Let us consider, $s = 1/2[\varphi^-(x_0 \ast y_0) + \max \{\varphi^-(x_0), \varphi^-(y_0)\}]$. We get that $s = 1/2(s + \max \{\eta_1, \eta_2\})$. Therefore, $\eta_1 < s_1 = 1/2(s + \max \{\eta_1, \eta_2\}) < s$ and $\eta_2 < s_2 = 1/2(s + \max \{\eta_1, \eta_2\}) < s$. Hence, $s > \max \{\eta_1, \eta_2\}$. Let us consider, $s = 1/2[\varphi^-(x_0 \ast y_0) + \max \{\varphi^-(x_0), \varphi^-(y_0)\}]$. We get that $s = 1/2(s + \max \{\eta_1, \eta_2\})$. Therefore, $\eta_1 < s_1 = 1/2(s + \max \{\eta_1, \eta_2\}) < s$ and $\eta_2 < s_2 = 1/2(s + \max \{\eta_1, \eta_2\}) < s$. Hence, $s > \max \{\eta_1, \eta_2\}$. Thus $s > \max \{\eta_1, \eta_2\}$. This implies $x_0, y_0 \in L(\varphi^-: s)$. Thus $\varphi^-(x \ast y) \leq \max \{\varphi^-(x), \varphi^-(y)\}$ for all $x, y \in X$. Hence, $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $BG$-subalgebra of $X$. 
Theorem 3.15. Any $BG$-subalgebra of $X$ can be realized as both the positive $t$-cut and negative $s$-cut of some bipolar fuzzy $BG$-subalgebra of $X$.

Proof. Let $P$ be a bipolar fuzzy $BG$-subalgebra of $X$ and $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy set on $X$ defined by

$$
\varphi^+(x) = \begin{cases} 
\lambda, & \text{if } x \in P \\
0, & \text{otherwise}
\end{cases}, \\
\varphi^-(x) = \begin{cases} 
\tau, & \text{if } x \in P \\
0, & \text{otherwise}
\end{cases}
$$

for all $\lambda \in [0,1]$ and $\tau \in [-1,0]$. We consider the following four cases:

Case (i). If $x, y \in P$, then $\varphi^+(x) = \lambda$, $\varphi^-(x) = \tau$ and $\varphi^+(y) = \lambda$ and $\varphi^-(y) = \tau$. Thus,

$$
\varphi^+(x \ast y) = \lambda = \min \{ \lambda, \lambda \} = \min \{ \varphi^+(x), \varphi^+(y) \} \\
\varphi^-(x \ast y) = \tau = \max \{ \tau, \tau \} = \max \{ \varphi^-(x), \varphi^-(y) \}.
$$

Case (ii). If $x \in P$ and $y \notin P$ then $\varphi^+(x) = \lambda$, $\varphi^-(x) = \tau$ and $\varphi^+(y) = 0$ and $\varphi^-(y) = 0$. Thus, $\varphi^+(x \ast y) \geq 0 = \min \{ \lambda, 0 \} = \min \{ \varphi^+(x), \varphi^+(y) \}$ and $\varphi^-(x \ast y) \leq 0 = \max \{ \tau, 0 \} = \max \{ \varphi^-(x), \varphi^-(y) \}$.

Case (iii). If $x \notin P$ and $y \in P$ then $\varphi^+(x) = 0$, $\varphi^-(x) = 0$ and $\varphi^+(y) = \lambda$ and $\varphi^-(y) = \tau$. Thus,

$$
\varphi^+(x \ast y) \geq 0 = \min \{ 0, \lambda \} = \min \{ \varphi^+(x), \varphi^+(y) \} \\
\varphi^-(x \ast y) \leq 0 = \max \{ 0, \tau \} = \max \{ \varphi^-(x), \varphi^-(y) \}.
$$

Case (iv). If $x \notin P$ and $y \notin P$ then $\varphi^+(x) = 0$, $\varphi^-(x) = 0$ and $\varphi^+(y) = 0$ and $\varphi^-(y) = 0$. Now $\varphi^+(x \ast y) \geq 0 = \min \{ 0, 0 \} = \min \{ \varphi^+(x), \varphi^+(y) \}$ and $\varphi^-(x \ast y) \leq 0 = \max \{ 0, 0 \} = \max \{ \varphi^-(x), \varphi^-(y) \}$.

Therefore $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $BG$-subalgebra of $X$.

Theorem 3.16. Let $P$ be a subset of $X$ and $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy set on $X$ which is given in the proof of Theorem 3.15. If the positive $t$-cut and negative $s$-cut of $\varphi$ are $BG$-subalgebras of $X$, then $P$ is a bipolar fuzzy $BG$-subalgebra of $X$.

Proof. Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy $BG$-subalgebra of $X$, and $x, y \in P$. Then $\varphi^+(x) = \lambda = \varphi^+(y)$ and $\varphi^-(x) = \tau = \varphi^-(y)$. Thus $\varphi^+(x \ast y) \geq 0 = \min \{ \varphi^+(x), \varphi^+(y) \}$

$$
= \min \{ \tau, \lambda \} = \lambda, \text{ and } \varphi^- (x \ast y) \leq 0 = \max \{ \varphi^-(x), \varphi^-(y) \} = \max \{ \tau, \tau \} = \tau,
$$

which imply that $x \ast y \in P$. Hence the theorem.

Theorem 3.17. If every bipolar fuzzy $BG$-subalgebra $\varphi = (\varphi^+, \varphi^-)$ of $X$ has the finite image, then every descending chain of $BG$-subalgebras of $X$ terminates at finite step.
Proof. Suppose there exists a strictly descending chain \( S_0 \supseteq S_1 \supseteq S_2 \cdots \) of \( BG \)-subalgebras of \( X \) which does not terminate at finite step. Define a bipolar fuzzy set \( \varphi = (\varphi^+, \varphi^-) \) in \( X \) by

\[
\varphi^+(x) = \begin{cases} 
\frac{n}{n+1} & \text{if } x \in S_n \setminus S_{n+1} \\
1 & \text{if } x \in \bigcap_{n=0}^{\infty} S_n 
\end{cases}
\]

and \( \varphi^-(x) = \begin{cases} 
-\frac{n}{n+1} & \text{if } x \in S_n \setminus S_{n+1} \\
-1 & \text{if } x \in \bigcap_{n=0}^{\infty} S_n 
\end{cases} \)

where \( n = 0, 1, 2, \cdots \) and \( S_0 \) stands for \( X \). Let \( x, y \in X \). Assume that \( x \in S_n \setminus S_{n+1} \) and \( y \in S_k \setminus S_{k+1} \) for \( n = 0, 1, 2, \cdots ; k = 0, 1, 2, \cdots \). Without loss of generality, we may assume that \( n \leq k \). Then obviously \( x \) and \( y \in S_n \) so \( x \ast y \in S_n \) because \( S_n \) is a \( BG \)-subalgebra of \( X \). Hence,

\[
\varphi^+(x \ast y) \geq \frac{n}{n+1} = \min \{\varphi^+(x), \varphi^+(y)\} \\
\varphi^-(x \ast y) \leq -\frac{n}{n+1} = \max \{\varphi^-(x), \varphi^-(y)\}.
\]

If \( x, y \in \bigcap_{n=0}^{\infty} S_n \), then \( x \ast y \in \bigcap_{n=0}^{\infty} S_n \). Thus

\[
\varphi^+(x \ast y) = 1 = \min \{\varphi^+(x), \varphi^+(y)\} \\
\varphi^-(x \ast y) = -1 = \min \{\varphi^-(x), \varphi^-(y)\}.
\]

If \( x \not\in \bigcap_{n=0}^{\infty} S_n \) and \( y \in \bigcap_{n=0}^{\infty} S_n \), then there exists a positive integer \( r \) such that \( x \in S_r \setminus S_{r+1} \). It follows that \( x \ast y \in S_r \) so that

\[
\varphi^+(x \ast y) \geq \frac{r}{r+1} = \min \{\varphi^+(x), \varphi^+(y)\} \\
\varphi^-(x \ast y) \leq -\frac{r}{r+1} = \max \{\varphi^-(x), \varphi^-(y)\}.
\]

Finally suppose that \( x \in \bigcap_{n=0}^{\infty} S_n \) and \( y \not\in \bigcap_{n=0}^{\infty} S_n \). Then \( y \in S_s \setminus S_{s+1} \) for some positive integer \( s \). It follows that \( x \ast y \in S_s \), and hence

\[
\varphi^+(x \ast y) \geq \frac{s}{s+1} = \min \{\varphi^+(x), \varphi^+(y)\} \\
\varphi^-(x \ast y) \leq -\frac{s}{s+1} = \max \{\varphi^-(x), \varphi^-(y)\}.
\]

This proves that \( \varphi = (\varphi^+, \varphi^-) \) is a bipolar fuzzy \( BG \)-subalgebras with an infinite number of different values, which is a contradiction. This completes the proof.

4. Conclusions and future work

In this paper, notion of bipolar fuzzy \( BG \)-subalgebras in \( BG \)-algebra are introduced and investigated some of their useful properties. It is my hope that this work would other foundations for further study of the theory of \( BG \)-algebras. In my future study of fuzzy structure of \( BG \)-algebra, may be the following topics should be considered: (i) to find bipolar fuzzy closed ideals in \( BG \)-algebra, (ii) to find the relationship between bipolar
fuzzy $BG$-subalgebras and closed ideals in $BG$-algebra, (iii) to find bipolar fuzzy translation in $BG$-algebra.

References


Singular Fuzzy Submodules

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Abstract:
In this paper we introduce the notion of singular fuzzy submodules of modules in terms of fuzzy essentiality. We investigate various characteristics of such submodules.

Keywords:
Fuzzy submodule, essential fuzzy submodule, fuzzy annihilators, singular submodule.

1. Introduction

In 1965, Lotfi Zadeh introduced the notion of fuzzy sets and since then this concept has been applied to many algebraic structures like groups, rings, modules, topologies and so on. Algebraic structures play an important role in Mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory and topological spaces and so on. Rosenfeld [16] was the first one to define the concept of fuzzy subgroups of a group. Since then many generalizations of this fundamental concept have been done (especially, in the last few decades). Naegoita and Ralescu [12] applied the concept of fuzzy set to the theory of modules and Pushkav [14] defined $t$-norm based fuzzy submodules. Pan [13] studied fuzzy finitely generated modules and quotient modules. Mukherjee et al [10], Kumar at al [4] studied various aspects of fuzzy submodules. In [19] Sidky introduced the notion of radical of a fuzzy submodule and also defined primary fuzzy submodule. Saikia and Kalita introduced the concept of fuzzy annihilators of fuzzy subsets of modules in [17] and consequently in [18] they have defined and characterized essential submodules. Recently, Rahman et al defined and characterized fuzzy small submodules, the dual notion of essential submodules in [15]. The notion of singularity plays a very important role in the study of algebraic structures. It was remarked by
Miguel Ferrero and Edmund R. Puczyłowski in [8] that studying properties of rings one can usually say more assuming that the considered rings are either singular or nonsingular. The same holds for modules. It has motivated us to study the fuzzy aspects of singularity. The concept of essentiality leads to the notion of singularity in rings and modules. In this paper using the concept of fuzzy essentiality we attempt to define singular fuzzy submodules and investigate different characteristics of such submodules.

2. Basic definitions and notations

Throughout this paper $R$ denotes a commutative ring with unity and $M$ denotes an $R$-module. The zero elements of $R$ and $M$ are 0 and $\theta$ respectively. The class of fuzzy subsets of $X$ is denoted by $[0,1]^X$.

Let $\mu \in [0,1]^R$. Then $\mu$ is called a fuzzy ideal of $R$ if it satisfies

(i) $\mu(x - y) \geq \mu(x) \wedge \mu(y) \forall x, y \in R$
(ii) $\mu(xy) \geq \mu(x) \vee \mu(y) \forall x, y \in R$.

The class of all fuzzy ideals of $R$ is denoted by $FI(R)$. Let $\mu \in [0,1]^M$. Then $\mu$ is called a fuzzy sub module of $M$ if it satisfies:

(i) $\mu(\theta) = 1$
(ii) $\mu(xy) \geq \mu(x) \wedge \mu(y) \forall x, y \in M$
(iii) $\mu(rx) \geq \mu(x)r \forall \in R, x \in M$.

The class of fuzzy sub modules is denoted by $FM$.

Let $\mu \in [0,1]^X$. Then a fuzzy point $x_i, x \in X, F \in (0,1]$ is defined as the fuzzy subset $x_i$ of $X$ such that $x_i(x) = t$ and $x_i(y) = 0$ for all $y \in X \setminus \{x\}$. We write $x_i \in \mu$.

Let $\mu \sigma \in [0,1]^M$. Then the sum of $\mu$ and $\sigma$ defined as

$$(\mu + \sigma)(x) = \{y \in \mu(y) \wedge \sigma(z) | y, z \in M, x = y + z \}.$$

Let $\mu \in [0,1]^M$ and $\sigma \in [0,1]^M$ then the product of $\mu$ and $\sigma$ is defined as

$$(\sigma \mu)(x) = \{\sigma(r) \wedge \mu(m) | r \in R, m \in M, x = rm \}.$$
Let \( \mu \in [0,1]^M \). Then annihilator of \( \mu \), denoted by \( \text{ann}(\mu) \), is defined as

\[
\text{ann}(\mu) = \bigcup \{ \eta \mid \eta \in [0,1]^R, \eta \mu \subseteq \chi_\theta \}.
\]

It is seen that \( \text{ann}(\mu) \subseteq [0,1]^R \).

Let \( a \in M \), \( a \neq \emptyset \) and \( \gamma \) be an essential fuzzy submodule of \( M \). We define a fuzzy set \( \sigma \in [0,1]^R \) by \( \sigma = \bigcup \{ \mu \mid \mu \in [0,1]^R, \mu a \subseteq \gamma \} \).

Let \( \delta \) be a fuzzy submodule of \( M \). Let us define a fuzzy subset \( Z(\delta) \) of \( M \) by \( Z(\delta) = \bigcup \{ \mu \mid \mu \in [0,1]^M, \mu \subseteq \delta, \mu \delta \subseteq \chi_\theta, \text{for some essential fuzzy ideal } \sigma \text{ of } R \} \).

3. Preliminaries

In this section we discuss some preliminary results needed for the sequel.

**Lemma 3.1.** If \( \mu \in [0,1]^M \) then

(i) \( \chi_\theta \subseteq \text{ann}(\mu) \).

(ii) \( \text{ann}(\mu) = \bigcup \{ r_\alpha \mid r \in R, \alpha \in [0,1], r_\alpha \mu \subseteq \chi_\theta \} \).

(iii) \( \mu \text{ann}(\mu) \subseteq \chi_\theta \).

**Lemma 3.2.** A submodule \( A \) is essential in \( M \) if and only if \( \chi_A \) is fuzzy essential in \( M \).

The above result also holds for essential ideal of rings. Thus, \( R \) being an essential ideal of \( R \), \( \chi_R \) is a fuzzy essential ideal in \( R \).

**Lemma 3.3.** \( Z(\delta) = \bigcup \{ m_\alpha \mid m_\alpha \in \delta, m_\alpha \sigma \subseteq \chi_\theta \}, \text{for some essential fuzzy ideal } \sigma \text{ of } R \} \).

**Proof.** Clearly, \( \{ m_\alpha \mid m_\alpha \in \delta, m_\alpha \sigma \subseteq \chi_\theta \}, \text{for some essential fuzzy ideal } \sigma \} \subseteq \{ \gamma \gamma \} \in [0,1]^M \), \( \gamma \subseteq \delta, \gamma \sigma \subseteq \chi_\theta \), for some essential fuzzy ideal \( \sigma \}. \) Therefore, \( \bigcup \{ m_\alpha \mid m_\alpha \in \delta, m_\alpha \sigma \subseteq \chi_\theta \}, \text{for some essential fuzzy ideal } \sigma \} \subseteq \bigcup \{ \gamma \gamma \} \in [0,1]^M \), \( \gamma \subseteq \delta, \gamma \sigma \subseteq \chi_\theta \), for some essential fuzzy ideal \( \sigma \} = Z(\delta) \).

Let \( \gamma \in [0,1]^M \) be such that \( \gamma \sigma \subseteq \chi_\theta \) for some essential fuzzy ideal \( \sigma \) of \( R \). Let \( m \in M \) be such that \( \gamma(m) = \alpha \). Now \( \langle m_\alpha \sigma \rangle(x) = \bigvee \{ m_\alpha(s) \wedge \sigma(y) \mid x = sy; s \in M, y \in R \} = \bigvee \{ \gamma(m) \wedge \sigma(z) \mid x = mz \} \leq \bigvee \{ \gamma(s) \wedge \sigma(y) \mid x = sy; s \in M, y \in R \} = (\gamma \sigma)(x) \leq \)
Thus \( m_{\sigma} \sigma \subseteq \chi_{\theta} \). So, \( Z(\delta) = \bigcup \left\{ \mu \mid \mu \in F(M), \mu \sigma \subseteq \chi_{\theta} \right\} \) for some essential fuzzy ideal \( \sigma \).

Hence we get the result. The following can be proved in a similar manner.

**Lemma 3.4.** \( Z(\delta) = \bigcup \left\{ \mu \mid \mu \in F(M), \mu \sigma \subseteq \chi_{\theta} \right\} \) for some essential fuzzy ideal \( \sigma \) of \( R \).

**Lemma 3.5.** Let \( \mu \) and \( \nu \) be fuzzy ideals of \( R \) such that \( \mu \subseteq \nu \) and \( \mu \) is an essential fuzzy ideal of \( R \). Then \( \nu \) is also an essential fuzzy ideal of \( R \).

### 4. Main results

We now present our main results

**Theorem 4.1.** \( Z(\delta) \) is a fuzzy submodule of \( M \).

*Proof.\* Clearly \( \chi_{\mathtt{\theta}} \) is an essential fuzzy ideal of \( R \). Since \( \chi_{\mathtt{\theta}} \chi_{\mathtt{\theta}} = \chi_{\mathtt{\theta}} \), so \( \chi_{\mathtt{\theta}} \subseteq Z(\delta) \).

Now \( Z(\delta)(m_{\phi}) \cap Z(\delta)(m_{\phi}) = (\bigvee \left\{ \gamma_{1}(m_{\phi}) \mid \gamma_{1} \subseteq \delta, \gamma_{1} \sigma_{1} \subseteq \chi_{\mathtt{\theta}} \right\} \) for some essential fuzzy ideal \( \sigma_{1} \) and \( \gamma_{2}(m_{\phi}) \mid \gamma_{2} \subseteq \delta, \gamma_{2} \sigma_{2} \subseteq \chi_{\mathtt{\theta}} \) for some essential fuzzy ideal \( \sigma_{2} \).

So, \( \gamma_{1} \sigma_{1} \subseteq \chi_{\mathtt{\theta}} \). Now \( \gamma_{1} \sigma_{1} \subseteq \chi_{\mathtt{\theta}} \), by Lemma 3.5, \( \gamma_{1} \sigma_{1} \) is also an essential fuzzy ideal of \( R \).

**Theorem 4.2.** For \( m_{\sigma} \in [0,1]^{M} \), \( m_{\sigma} \in Z(\delta) \) if and only if \( \text{ann}(m_{\sigma}) \) is an essential fuzzy ideal of \( R \).

*Proof.\* Let \( m_{\sigma} \in Z(\delta) \). Then \( m_{\sigma} \sigma \subseteq \chi_{\mathtt{\theta}} \), for some essential fuzzy ideal \( \sigma \) of \( R \).

So, \( \sigma \subseteq \text{ann}(m_{\sigma}) \). Now \( \sigma \) being an essential fuzzy ideal of \( R \), by Lemma 3.5, \( \text{ann}(m_{\sigma}) \) is also an essential fuzzy ideal of \( R \).

Conversely, let \( \text{ann}(m_{\sigma}) \) be an essential fuzzy ideal of \( R \). Now, by Lemma 3.1, \( m_{\sigma} \text{ann}(m_{\sigma}) \subseteq \chi_{\mathtt{\theta}} \). By definition of \( Z(\delta) \), \( m_{\sigma} \subseteq Z(\delta) \).
Theorem 4.3. If $\delta_1$ and $\delta_2$ are fuzzy submodules of $M$ with $\delta_2 \subseteq \delta_1$ then $Z(\delta_2) = \delta_2 \cap Z(\delta_1)$.

Proof. Let $m_a \in Z(\delta_2)$. Then $m_a \in \delta_2$ and $m_a \sigma \subseteq \chi_\theta$ for some essential fuzzy ideal $\sigma$ of $R$. So $m_a \in \delta_1$ and $m_a \sigma \subseteq \chi_\theta$ imply $m_a \in Z(\delta_1)$. Therefore $m_a \in \delta_2 \cap Z(\delta_1)$.

Conversely let $m_a \in Z(\delta_1) \cap \delta_2$. Then $m_a \in \delta_2$ and $m_a \sigma \subseteq \chi_\theta$ for some essential fuzzy ideal $\sigma$. So $m_a \in Z(\delta_1)$. Consequently $Z(\delta_1) = \delta_1 \cap Z(\delta_1)$.

Definition 4.4. A fuzzy submodule $\delta$ of $M$ is singular or non singular according as $Z(\delta) = \delta$ or $\chi_\theta$.

Lemma 4.5. If $\delta \in F(M)$, then $Z(\delta)$ is singular.

Proof. $Z(\delta) = \bigcup \{m_a | m_a \in Z(\delta), m_a \sigma \subseteq \chi_\theta \text{ for some essential fuzzy ideal } \sigma\}$

$= \bigcup \{m_a | m_a \in Z(\delta)\} = Z(\delta)$

Theorem 4.6. (1) Let $\delta_2$ be any fuzzy submodule of a non singular (singular) fuzzy submodule $\delta_1$. Then $\delta_2$ is also non singular (singular).

(2) If $\delta_1$ is an essential extension of a non singular fuzzy submodule $\delta_2$, $\delta_1$ is non singular.

Proof. (1) By Theorem 4.3, $Z(\delta_2) = \delta_2 \cap Z(\delta_1)$. $\delta_1$ being non singular implies $Z(\delta_1) = \chi_\theta$. So $Z(\delta_2) = \delta_2 \cap \chi_\theta = \chi_\theta$. Therefore $\delta_2$ is also non singular. Again $\delta_1$ is singular implies $Z(\delta_1) = \delta_1$. So $Z(\delta_2) = \delta_2 \cap \delta_1 = \delta_1$. Therefore $\delta_2$ is singular.

(2) $\delta_1$ non singular implies $Z(\delta_1) = \chi_\theta$.

Now $Z(\delta_2) = \delta_2 \cap Z(\delta_1)$ implies $\delta_2 \cap Z(\delta_1) = \chi_\theta$. But $\delta_1$ is an essential extension of $\delta_2$. So $Z(\delta_1) = \chi_\theta$. Therefore $\delta_1$ is non singular.

Theorem 4.7. The sum (direct or not) of two singular fuzzy submodules of a fuzzy submodule is again singular.

Proof. Let $\delta_1$ and $\delta_2$ be two singular fuzzy submodules such that $\delta_1 \subseteq \delta$ and $\delta_2 \subseteq \delta$. Then $Z(\delta_1) = \delta_1$, $Z(\delta_2) = \delta_2$. Now $\delta_1 \subseteq \delta \Rightarrow Z(\delta_1) \subseteq Z(\delta) \Rightarrow \delta_1 \subseteq Z(\delta)$ and $\delta_2 \subseteq \delta \Rightarrow Z(\delta_2) \subseteq Z(\delta) \Rightarrow \delta_2 \subseteq Z(\delta)$.

$\therefore \delta_1 + \delta_2 \subseteq Z(\delta)$ By Lemma 4.5, $Z(\delta)$ is singular. So $\delta_1 + \delta_2$ is a fuzzy submodule of a singular fuzzy submodule $Z(\delta)$. Hence by Theorem 4.6, $\delta_1 + \delta_2$ is singular.

5. Conclusions
The study of fuzzy substructures of algebraic structures has attracted many researchers in the last few decades. In our attempt to investigate rings with finiteness conditions, we have introduced essential fuzzy ideals and annihilators of fuzzy subsets. In this paper these two notions help us to develop the concept of singularity which in turn will lead us to many aspects of modules with chain conditions on various fuzzy substructures.

References

Abstract:
This paper introduces a new class of sets called generalized regular fuzzy closed sets which is a stronger form of generalized fuzzy closed sets. Basic properties of generalized regular fuzzy closed sets are analyzed. Generalized fuzzy closed sets lie between generalized regular fuzzy closed sets and regular generalized fuzzy closed sets. Generally fuzzy closed set is not a generalized regular fuzzy closed set. But in this paper it is shown that closure of any fuzzy open set is always a generalized regular fuzzy closed set. With the help of generalized regular fuzzy closed sets, the concept of generalized regular fuzzy continuity and fuzzy generalized regular closed irresolute mapping are introduced. Latter on the interrelationship between generalized regular fuzzy continuity, generalized fuzzy continuity and regular generalized fuzzy continuity are also discussed. Different kinds of contra continuity based on fuzzy closed set, regular generalized fuzzy closed set, generalized regular fuzzy closed set are defined and few related results are shown. The decomposition of fuzzy homeomorphism is discussed via generalized fuzzy homeomorphism, generalized regular fuzzy homeomorphism and regular generalized fuzzy homeomorphism.

Key words:

1. Introduction and preliminaries

Jin Han Park and Jin Keun Park [8] introduced weaker form of generalized fuzzy closed set and generalized fuzzy continuous mapping i.e. regular generalized fuzzy closed set and generalizations of fuzzy continuous functions. In this paper we define and study another generalization of fuzzy closed set i.e. generalized regular fuzzy closed set which is the stronger form of the previous two generalizations. Section 2 is devoted to generalize regular fuzzy closed sets and their properties. Section 3 is contributed to generalize regular fuzzy open sets and their properties. In Section 4 we introduce generalized regular fuzzy continuous functions and fuzzy generalized regular closed irresolute functions and investigate interrelation between them. In Section 5 we introduce generalized regular fuzzy contra continuity. Lastly in Section 6 we introduce generalized regular fuzzy homeomorphism, regular generalized fuzzy homeomorphism, generalized regular fuzzy homeomorphism, generalized fuzzy closed homeomorphism, regular generalized fuzzy closed homeomorphism and generalized regular fuzzy closed homeomorphism.

Throughout this paper, simply by $X$ and $Y$ we shall denote fts's $(X, \tau)$ and $(Y, \sigma)$ and $f : X \to Y$ will mean that $f$ is a function from $(X, \tau)$ to $(Y, \sigma)$. For a fuzzy set $A$ of $X$, $\text{cl}(A)$, $\text{int}(A)$ and $1 - A$ will denote the closure of $A$, the interior of $A$ and the complement of $A$ respectively, whereas the constant fuzzy sets taking on the values 0 and 1 on $X$ are denoted by $\mathbf{0}_X$ and $\mathbf{1}_X$ respectively. A fuzzy set $A$ of a fts is called fuzzy regular open [1], if $A = \text{int}(\text{cl}(A))$. For any fuzzy subset $A$, $\text{RCI}(A) = \bigwedge \{G : A \subseteq G, G$ is a fuzzy regular closed subset of $X\}$. Before entering into our work, we recall the following definitions which are prerequisite for this paper.

**Definition 1.1** [1]. A fuzzy set $\lambda$ of a fuzzy space $X$ is called a fuzzy regular closed set of $X$ if $\text{Cl}(\lambda) = \lambda$.

**Definition 1.2** [2]. Let $X$ be a fuzzy topological space. A fuzzy set $\lambda$ in $X$ is called generalized fuzzy closed set (in short, $gf$-closed) if $\text{cl}(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and $\mu$ is fuzzy open.

**Definition 1.3** [8]. A fuzzy set $A$ in a fts $X$ is called regular generalized fuzzy closed (in short, $rgf$-closed) if $\text{cl}(A) \leq \mu$ whenever $A \subseteq U$ and $U$ is fuzzy regular open. A fuzzy set $A$ is called regular generalized fuzzy open (in short, $rgf$-open) if its compliment $1 - A$ is $rgf$-closed.

**Definition 1.4** [2]. A function $f : X \to Y$ is called generalized fuzzy continuous (in short, $gf$-continuous) if the inverse image of every fuzzy closed set in $Y$ is $gf$-closed in $X$. 

Definition 1.5 [8]. A function $f : X \to Y$ is called regular generalized fuzzy continuous (in short $rgf$-continuous) if the inverse image of every fuzzy closed set in $Y$ is $rgf$-closed in $X$.

Definition 1.6 [2]. A function $f : (X, \tau) \to (Y, \sigma)$ is called fuzzy generalized closed irresolute (in short $fgc$-irresolute) if $f^{-1}(V)$ is generalized fuzzy closed in $(X, \tau)$ for every generalized fuzzy closed set $V$ in $(Y, \sigma)$.

Definition 1.7 [8]. A function $f : (X, \tau) \to (Y, \sigma)$ is called fuzzy regular generalized closed irresolute (in short $frgc$-irresolute) if $f^{-1}(V)$ is regular generalized fuzzy closed in $(X, \tau)$ for every regular generalized fuzzy closed set $V$ in $(Y, \sigma)$.

Definition 1.8 [1]. A mapping $f : (X, \tau) \to (Y, \tau')$ form a fuzzy space $X$ to another fuzzy space $Y$ is called fuzzy almost continuous mapping if $f^{-1}(\lambda) \in \tau X$ for each fuzzy regular open set $\lambda$ of $Y$.

2. Generalized regular fuzzy closed sets

Definition 2.1. A fuzzy set $A$ in a fts $X$ is called generalized regular fuzzy closed set (in short, $grf$-closed) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is fuzzy open.

Theorem 2.2. If $A$ and $B$ are $grf$-closed sets then $A \cup B$ is also $grf$-closed set.

Proof. Let $A \cup B \subseteq U$ and $U$ be fuzzy open. Then $A, B \subseteq U$ and thus $\text{cl}(A) \subseteq U, \text{cl}(B) \subseteq U$. Hence $\text{cl}(A \cup B) \subseteq U$ from the fact that $\text{cl}(A) \cup \text{cl}(B) = \text{cl}(A \cup B)$.

Remark 2.3. Intersection of any two $grf$-closed sets need not be a $grf$-closed set.

Example 2.4. Let $X = \{a, b, c\}$, $\tau_1 = \{0_X, 1_X, A_1\}$ where $A_1(a) = 0.4$, $A_1(b) = 0.5$, $A_1(c) = 0.6$. $A_2(a) = 0.4$, $A_2(b) = 0.7$, $A_2(c) = 0.6$, $A_3(a) = 0.9$, $A_3(b) = 0.5$, $A_3(c) = 0.8$. Here $A_1$ and $A_2$ are two $grf$-closed sets but $A_1 \cap A_2$ is not $grf$-closed set.

Proposition 2.5. Every fuzzy regular closed set is $grf$-closed set.

Proof. It is obvious. But the converse is not true as shown in the following example.

Example 2.6. Let $X = \{a, b, c\}$, $\tau_1 = \{0_X, 1_X, A_1\}$ where $A_1(a) = 0.4$, $A_1(b) = 0.3$, $A_1(c) = 0.6$. $A_2(a) = 0.4$, $A_2(b) = 0.5$, $A_2(c) = 0.3$. Here $A_2$ is not fuzzy regular closed though it is $grf$-closed set.
Proposition 2.7. Every grf-closed set is rgf-closed set.

Proof. It is obvious from the fact that \( \text{cl}(A) \leq R\text{cl}(A) \). But the converse may not be true as shown in the following example.

Example 2.8. Let \( X = \{a, b, c\} \), \( \tau = \{0_x, 1_x, A_i\} \), where \( A_i(a) = 0.4 \), \( A_i(b) = 0.7 \), \( A_i(c) = 0.3 \). The fuzzy set \( A_2 \) defined as \( A_2(a) = 0.3 \), \( A_2(b) = 0.3 \), \( A_2(c) = 0.2 \) is rgf-closed set but not grf-closed set.

Proposition 2.9. Every grf-closed set is fuzzy generalized closed set.

Proof. Let \( A \subseteq X \) be grf-closed set. Let \( U \) be fuzzy open. So \( R\text{cl}(A) \leq U \). Since every fuzzy regular closed set is fuzzy closed set so \( \text{cl}(A) \leq R\text{cl}(A) \leq U \). But the converse is not true in general as shown in the following example.

Example 2.10. Let \( X = \{a, b, c\} \), \( \tau = \{0_x, 1_x, A_i\} \) where \( A_i(a) = 0.5 \), \( A_i(b) = 0.7 \), \( A_i(c) = 0.9 \). Then \( 1 - A_i \) is fgc set but it is not grf-closed set.

Theorem 2.11. If \( A \) be any fuzzy open and grf-closed set, then \( A \) is fuzzy regular closed and hence fuzzy clopen.

Proof. If \( A \) is fuzzy open and grf-closed set then \( R\text{cl}(A) \leq A \). This implies \( A \) is regular closed. Hence \( A \) is fuzzy clopen.

Theorem 2.12. If \( A \) is fuzzy preclosed and fuzzy semi-open then it is grf-closed set.

Proof. If \( A \) is fuzzy preclosed then \( \text{cl}(\text{int}(A)) \leq A \) and also if \( A \) is fuzzy semi-open then \( A \leq \text{cl}(\text{int}(A)) \). Both of this holds when \( A \) is fuzzy regular closed. It implies that \( A \) is grf-closed set.

Theorem 2.13. If \( A \) and \( B \) be any two fuzzy regular closed subsets of \( (X, \tau) \) then \( A \lor B \) is also grf-closed subsets of \( (X, \tau) \).

Proof. If \( A \) and \( B \) be any two fuzzy regular closed subsets of \( (X, \tau) \) then \( A \lor B \) is also fuzzy regular closed subsets of \( (X, \tau) \). And every fuzzy regular closed set is grf-closed et. So \( A \lor B \) is grf-closed subsets of \( (X, \tau) \).


Proof. In [1] it is shown that if \( \lambda \) be a fuzzy open set of a fuzzy space \( X \) then \( \text{int}(\text{cl}(\lambda)) \leq \text{cl}(\lambda) \) implies that \( \text{cl}(\text{int}(\text{cl}(\lambda))) \leq \text{cl}(\lambda) \). Now \( \lambda \) is fuzzy open implies that \( \lambda \leq \text{int}(\text{cl}(A)) \) and hence \( \text{cl}(\lambda) \leq \text{cl}(\text{int}(\text{cl}(\lambda))) \). Thus \( \text{cl}(\lambda) \) is fuzzy regular.
closed set. And every fuzzy regular closed set is \( grf \)-closed set. Therefore the closure of a fuzzy open set is \( grf \)-closed set.

**Remark 2.15.** If \( A \) be any \( gfc \) set in \( (X, \tau) \) and if \( cl(A) \leq Rcl(A) \leq U \) where \( U \) is open then \( A \) is \( grf \)-closed set.

**Theorem 2.16.** If \( A \) is \( grf \)-closed set and \( A \leq B \leq Rcl(A) \) then \( B \) is \( grf \)-closed set.

**Proof.** Let \( B \leq U \) where \( U \) is open. Since \( A \leq B \), therefore \( A \leq U \) and \( A \) is \( grf \)-closed \( Rcl(A) \leq U \). But \( Rcl(B) \leq Rcl(A) \) since \( B \leq Rcl(A) \) and also \( Rcl(B) \leq U \). Hence \( B \) is \( grf \)-closed set.

### 3. Generalized regular fuzzy open sets

**Definition 3.1.** A fuzzy set \( A \) in a fts \( X \) is called generalized regular fuzzy open set (in short, \( grf \)-open) iff its compliment is generalized regular fuzzy closed.

**Theorem 3.2.** A fuzzy set \( A \) is \( grf \)-open iff \( F \leq R\text{int}(A) \) whenever \( F \) is fuzzy closed and \( F \leq A \).

**Proof.** Let \( A \) be \( grf \)-open set and \( F \) be a fuzzy closed set such that \( F \leq A \). Then \( 1-A \leq 1-F \). Where \( 1-F \) is open. Generalized regular closedness of \( 1-A \) implies that \( 1-R\text{int}A = Rcl(1-A) \leq 1-F \) which implies \( F \leq R\text{int}(A) \).

Conversely, suppose that \( A \) is a fuzzy set such that \( F \leq R\text{int}(A) \) whenever \( F \) is closed and \( F \leq A \). We claim that \( 1-A \) is \( grf \)-closed. For if \( 1-A \leq U \) where \( U \) is fuzzy open then \( 1-A \leq U \), \( 1-U \leq A \). Hence by assumption \( 1-U \leq R\text{int}(A) \). i.e. \( 1-R\text{int}(A) \leq U \). Hence \( Rcl(1-A) \leq U \) which implies \( 1-A \) is generalized regular fuzzy closed set. Therefore \( A \) is \( grf \)-open set.

**Theorem 3.3.** If \( R\text{int}(A) \leq B \leq A \) and \( A \) is \( grf \)-open then \( B \) is \( grf \)-open.

**Proof.** \( R\text{int}(A) \leq B \leq A \) implies \( 1-A \leq 1-B \leq 1-R\text{int}(A) \). That is \( 1-A \leq 1-B \leq Rcl(1-A) \). Since \( 1-A \) is \( grf \)-closed, by Theorem 2.16, \( 1-B \) is \( grf \)-closed and \( B \) is \( grf \)-open.

**Remark 3.4.** (i) The union of two \( grf \)-open sets is not generally \( grf \)-open. Example 2.5 serves the purpose.
(ii) The intersection of any two \( grf \)-open sets is \( grf \)-open.

From the discussion we get the following relations
4. Generalized regular fuzzy continuous functions and fuzzy generalized regular closed irresolute functions

Definition 4.1. A function \( f : X \to Y \) is called generalized regular fuzzy continuous (in short, grf continuous) if the inverse image of every fuzzy closed set in \( Y \) is grf-closed in \( X \).

Theorem 4.2. Every generalized regular fuzzy continuous function is generalized fuzzy continuous.

Proof. It is obvious.

Remark 4.3. However the converse of the above theorem may not true be as shown in the following example.

Example 4.4. \( X = \{a, b, c\} = Y \), \( \tau_1 = \{0, 1, A_1\} \) where \( A_1(a) = 0.5 \), \( A_1(b) = 0.7 \), \( A_1(c) = 0.4 \), \( \tau_2 = \{0, 1, A_2\} \) where \( A_2(a) = 0.7 \), \( A_2(b) = 0.5 \), \( A_2(c) = 0.9 \). Here \( f : X \to Y \) defined by \( f(a) = b \), \( f(b) = a \), \( f(c) = c \). Here the inverse image of fuzzy closed set is generalized fuzzy closed set. Therefore it is generalized fuzzy continuous function. But the inverse image of fuzzy closed set \( \{(a, 0.3), (b, 0.5), (c, 0.1)\} \) is not generalized regular fuzzy closed set. Thus \( f \) is not generalized regular fuzzy continuous function.

Theorem 4.5. Every generalized regular fuzzy continuous function is regular generalized fuzzy continuous function.

Remark 4.6. However the converse of the above theorem is not true as shown in the following example.

Example 4.7. Let \( X = \{a, b, c\} = Y \), \( \tau_1 = \{0, 1, A_1\} \) where \( A_1(a) = 0.5 \), \( A_1(b) = 0.7 \), \( A_1(c) = 0.6 \). And \( \tau_2 = \{0, 1, A_2\} \) where \( A_2(a) = 0.7 \), \( A_2(b) = 0.6 \), \( A_2(c) = 0.5 \). Here \( f : X \to Y \) defined by \( f(a) = b \), \( f(b) = a \), \( f(c) = c \). Then \( f \) is regular generalized fuzzy continuous function but not generalized regular fuzzy continuous function. Since \( f^{-1}(1 - A_2) \) is not generalized regular fuzzy closed in \( (X, \tau_1) \) for a fuzzy closed set \( (1 - A_1) \) in \( (Y, \tau_2) \).
Proposition 4.8. A function \( f : X \to Y \) is generalized regular fuzzy continuous as well as fuzzy continuous if the inverse image of fuzzy closed set is regular fuzzy closed set.

Theorem 4.9. Let \( f : (X, \tau) \to (Y, \sigma) \) be a function. Then following statements are equivalent:

(i) \( f \) is generalized regular fuzzy continuous.

(ii) The inverse image of each fuzzy open set in \( Y \) is grf-open in \( X \).

Theorem 4.10. Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be any two function. If \( f \) is generalized regular fuzzy continuous and \( g \) is fuzzy continuous then the composition \( gof \) is generalized regular fuzzy continuous.

Proof. Let \( V \) be a fuzzy closed set in \( (Z, \eta) \). Then \( g^{-1}(V) \) is closed in \( (Y, \sigma) \) as \( g \) is continuous. Generalized regular fuzzy continuity of \( f \) implies that \( f^{-1}(g^{-1}(V)) \) is generalized regular fuzzy closed set in \( (X, \tau) \). That is, \( (gof)^{-1}(V) \) is generalized regular fuzzy closed in \( (X, \tau) \). Hence \( gof \) is generalized regular fuzzy continuous.

Remark 4.11. The composition of two generalized regular fuzzy continuous functions need not be a generalized regular fuzzy continuous function.

Example 4.12. Let \( X = Y = Z = \{a, b, c\} \), \( \tau_1 = \{0_x, 1_x, A_1\} \) where \( A_1(a) = 0.5 \), \( A_1(b) = 0.7 \), \( A_1(c) = 0.4 \), \( \tau_2 = \{0_y, 1_y, A_2\} \) where \( A_2(a) = 0.5 \), \( A_2(b) = 0.5 \), \( A_2(c) = 0.5 \) and \( \tau_3 = \{0_z, 1_z, A_3\} \) where \( A_3(a) = 0.5 \), \( A_3(b) = 0.3 \), \( A_3(c) = 0.6 \). For the identity functions \( f : X \to Y \) and \( g : Y \to Z \), \( (gof)^{-1}(1 - A_1) \) is not generalized regular fuzzy closed set in \( (X, \tau_1) \) for fuzzy closed set \( (1 - A_1) \) in \( (Z, \tau_3) \).

Definition 4.13. A function \( f : (X, \tau) \to (Y, \sigma) \) is called fuzzy regular irresolute if \( f^{-1}(V) \) is fuzzy regular open (resp. closed) in \( (X, \tau) \) for every fuzzy regular open (resp. closed) set \( V \) in \( (Y, \sigma) \).

Definition 4.14. A function \( f : (X, \tau) \to (Y, \sigma) \) is called fuzzy generalized regular closed irresolute (in short \( fgrc \)- irresolute) if \( f^{-1}(V) \) is generalized regular fuzzy closed in \( (X, \tau) \) for every generalized regular fuzzy closed set \( V \) in \( (Y, \sigma) \).

Remark 4.15. Fuzzy generalized regular closed irresolute function and generalized regular fuzzy continuous function are independent of each other concept.

Example 4.16. Generalized regular fuzzy continuous function does not imply fuzzy generalized regular closed irresolute function.
Let \( X = Y = \{ a, b, c \} \), \( \tau_1 = \{ 0, 1, A_1 \} \) where \( A_1(a) = 0.4, A_1(b) = 0.5, A_1(c) = 0.7 \) and \( \tau_2 = \{ 0, 1, A_2 \} \) where \( A_2(a) = 0.4, A_2(b) = 0.5, A_2(c) = 0.6 \). The identity function \( f \) is generalized regular fuzzy continuous but not fuzzy generalized regular closed irresolute function. For a fuzzy set \( A_3 \) in \( Y \) defined by \( A_3(a) = 0.5, A_3(b) = 0.4, A_3(c) = 0.7 \) is generalized regular fuzzy closed in \( (Y, \tau_1) \). \( f^{-1}(A_3) \) is not generalized regular fuzzy closed in \( (X, \tau_1) \).

**Example 4.17.** Fuzzy generalized regular closed irresolute function may not be a generalized regular fuzzy continuous function.

Let \( X = Y = \{ a, b, c \} \), \( \tau_1 = \{ 0, 1, A_1 \} \) where \( A_1(a) = 0.4, A_1(b) = 0.6, A_1(c) = 0.8 \) and \( \tau_2 = \{ 0, 1, A_2 \} \) where \( A_2(a) = 0.5, A_2(b) = 0.7, A_2(c) = 0.9 \). Here \( f : X \to Y \) defined by \( f(a) = b, f(b) = a, f(c) = c \), is generalized regular fuzzy closed irresolute function but not generalized regular fuzzy continuous function as \( f^{-1}(1 - A_1) \) is not generalized regular fuzzy closed set in \( (X, \tau_1) \).

**Remark 4.18.** Fuzzy generalized regular closed irresolute function and fuzzy generalized closed irresolute function are independent of each other.

**Example 4.19.** Fuzzy generalized regular closed irresolute function may not be a fuzzy generalized closed irresolute function.

Let \( X = Y = \{ a, b, c \} \), \( \tau_1 = \{ 0, 1, A_1 \} \) where \( A_1(a) = 0.4, A_1(b) = 0.6, A_1(c) = 0.8 \). \( \tau_2 = \{ 0, 1, A_2 \} \) where \( A_2(a) = 0.5, A_2(b) = 0.7, A_2(c) = 0.9 \). Here \( f : X \to Y \) defined by \( f(a) = b, f(b) = a, f(c) = c \). Then \( f \) is fuzzy generalized regular closed irresolute function but not fuzzy generalized closed irresolute function. For a fuzzy set \( 1 - A_2 \) in \( Y \) defined by \( A_3(a) = 0.5, A_3(b) = 0.3, A_3(c) = 0.1 \) is generalized fuzzy closed in \( (Y, \tau_1) \). \( f^{-1}(A_3) \) is not generalized fuzzy closed in \( (X, \tau_1) \).

**Example 4.20.** Fuzzy generalized closed irresolute function may not be a fuzzy generalized regular closed irresolute function.

Let \( X = Y = \{ a, b, c \} \), \( \tau_1 = \{ 0, 1, A_1 \} \) where \( A_1(a) = 0.5, A_1(b) = 0.4, A_1(c) = 0.3 \). \( \tau_2 = \{ 0, 1, A_2 \} \) where \( A_2(a) = 0.6, A_2(b) = 0.5, A_2(c) = 0.7 \). Here \( f : X \to Y \) defined by \( f(a) = b, f(b) = a, f(c) = c \). Then \( f \) is fuzzy generalized closed irresolute function but not fuzzy generalized regular closed irresolute function. For a fuzzy set \( A_1 \) in \( Y \) defined by \( A_1(a) = 0.4, A_1(b) = 0.5, A_1(c) = 0.3 \) is generalized regular fuzzy closed in \( (Y, \tau_1) \). \( f^{-1}(A_1) \) is not generalized regular fuzzy closed in \( (X, \tau_1) \).

**Theorem 4.21.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a function. Then following statements are equivalent:

(i) \( f \) is \( fgrc \)-irresolute.
(ii) The inverse image of each $grf$-open set in $Y$ is $grf$-open in $X$.

**Theorem 4.22.** Let $f : X \to Y$ and $g : Y \to Z$ be function. If $f$ and $g$ both are $fgrc$-irresolute then the composition $gof$ is also $fgrc$-irresolute.

**Definition 4.23.** A mapping $f : (X, \tau) \to (Y, \sigma)$ form a fuzzy space $X$ to another fuzzy space $Y$ is called generalized regular fuzzy almost continuous mapping if $f^{-1}(\lambda)$ is generalized regular fuzzy closed in $X$ for each fuzzy regular open set $\lambda$ of $Y$.

**Theorem 4.24.** Every generalized regular fuzzy continuous function is generalized regular fuzzy almost continuous function.

**Remark 4.25.** However the converse of the above theorem is not true as shown in the following example.

**Example 4.26.** Let $X = \{a, b, c\} = Y$, $\tau = \{0_x, 1_x, A_x\}$ where $A_x(a) = 0.5$, $A_x(b) = 0.7$, $A_x(c) = 0.6$. $\tau_2 = \{0_y, 1_y, A_y\}$ where $A_y(a) = 0.7$, $A_y(b) = 0.6$, $A_y(c) = 0.5$. Here $f : X \to Y$ defined by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then $f$ is generalized regular fuzzy almost continuous function but not generalized regular fuzzy continuous function. Since $f^{-1}(1 - A_2)$ is not generalized regular fuzzy closed in $(X, \tau)$ for a fuzzy closed set $(1 - A_2)$ in $(Y, \tau_2)$.

5. **Generalized regular fuzzy contra continuous functions**

**Definition 5.1.** A function $f : X \to Y$ is called generalized fuzzy contra continuous if the inverse image of every fuzzy open set in $Y$ is $gfc$ (generalized fuzzy closed) set in $X$.

**Definition 5.2.** A function $f : X \to Y$ is called generalized regular fuzzy contra continuous if the inverse image of every fuzzy open set in $Y$ is $grf$-closed set in $X$. 

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Fuzzy
continuity

Generalized regular fuzzy
continuity

Generalized fuzzy
continuity

Regular generalized fuzzy
continuity
Definition 5.3. A function \( f : X \to Y \) is called regular generalized fuzzy contra continuous if the inverse image of every fuzzy open set in \( Y \) is \( rgf \)-closed (regular generalized fuzzy closed) set in \( X \).

Theorem 5.4. Every generalized regular fuzzy contra continuous function is generalized fuzzy contra continuous.

Remark 5.5. However the converse of the above theorem may not be true as shown in the following example.

Example 5.6. \( X = \{a, b, c\} = Y \), \( \tau_1 = \{0_\times, 1_\times, A_1\} \) where \( A_1(a) = 0.6 \), \( A_1(b) = 0.7 \), \( A_1(c) = 0.5 \), \( \tau_2 = \{0_\times, 1_\times, A_2\} \) where \( A_2(a) = 0.4 \), \( A_2(b) = 0.3 \), \( A_2(c) = 0.5 \). Here \( f : X \to Y \) defined by \( f(a) = a \), \( f(b) = b \), \( f(c) = c \). Here the inverse image of fuzzy open set in \( Y \) is generalized closed in \( X \). Therefore it is generalized fuzzy contra continuous. But the inverse image of fuzzy open set \( \{(a, 0.4), (b, 0.3), (c, 0.5)\} \) in \( Y \) is not generalized regular fuzzy closed set in \( X \). Here \( f \) is not generalized regular fuzzy contra continuous function.

Theorem 5.7. Every generalized fuzzy contra continuous is regular generalized fuzzy contra continuous function.

Remark 5.8. However the converse of the above theorem may not be true as shown in the following example.

Example 5.9. \( X = \{a, b, c\} = Y \), \( \tau_1 = \{0_\times, 1_\times, A_1\} \) where \( A_1(a) = 0.4 \), \( A_1(b) = 0.6 \), \( A_1(c) = 0.7 \). \( \tau_2 = \{0_\times, 1_\times, A_2\} \) where \( A_2(a) = 0.2 \), \( A_2(b) = 0.5 \), \( A_2(c) = 0.4 \). Here \( f : X \to Y \) defined by \( f(a) = a \), \( f(b) = b \), \( f(c) = c \). Here the inverse image of fuzzy open set in \( Y \) is regular generalized closed in \( X \). Therefore it is regular generalized fuzzy contra continuous. But the inverse image of fuzzy open set \( \{(a, 0.2), (b, 0.5), (c, 0.4)\} \) in \( Y \) is not generalized fuzzy closed set in \( X \). Here \( f \) is not generalized fuzzy contra continuous function.

Theorem 5.10. Let \( f : X \to Y \) and \( g : Y \to Z \) be function.

(i) If \( f \) is fuzzy generalized closed irresolute and \( g \) is generalized fuzzy contra continuous then \( gof \) is generalized fuzzy contra continuous function.

(ii) If \( f \) is fuzzy generalized regular closed irresolute and \( g \) is generalized regular fuzzy contra continuous then \( gof \) is generalized regular fuzzy contra continuous function.

(iii) If \( f \) is fuzzy regular generalized closed irresolute and \( g \) is regular generalized fuzzy contra continuous then \( gof \) is regular generalized fuzzy contra continuous function.
6. Generalized regular fuzzy homeomorphisms

**Definition 6.1.** A bijection \( f : X \to Y \) is called generalized fuzzy homeomorphism if \( f \) is generalized fuzzy continuous and generalized fuzzy open.

**Definition 6.2.** A bijection \( f : X \to Y \) is called regular generalized fuzzy homeomorphism if \( f \) is regular generalized fuzzy continuous and regular generalized fuzzy open.

**Definition 6.3.** A bijection \( f : X \to Y \) is called generalized regular fuzzy homeomorphism if \( f \) is generalized regular fuzzy continuous and generalized regular fuzzy open.

**Theorem 6.4.** Every fuzzy homeomorphism is generalized fuzzy homeomorphism.

**Proof.** It is obvious.

**Remark 6.5.** The converse of the above theorem may not be true in general as shown in the following example.

**Example 6.6.** Let \( X = \{a, b, c\} \), \( Y = \{p, q, r\} \), \( \tau_1 = \{0 \_X, 1 \_X, \lambda\} \) where \( \lambda(a) = 1 \), \( \lambda(b) = 0 \), \( \lambda(c) = 0 \) and \( \tau_2 = \{0 \_Y, 1 \_Y, \mu\} \) where \( \mu(p) = 0 \), \( \mu(q) = 1 \), \( \mu(r) = 0 \). Here \( f : X \to Y \) defined by \( f(a) = p \), \( f(b) = q \), \( f(c) = r \). Then \( f \) is generalized fuzzy homeomorphism but \( f \) is not fuzzy homeomorphism.

**Theorem 6.7.** Every generalized fuzzy homeomorphism is regular generalized fuzzy homeomorphism.

**Proof.** By the definition, it can be proved.

**Remark 6.8.** The converse of the above theorem may not be true as shown in the following example.

**Example 6.9.** Let \( X = \{a, b, c\} \), \( Y = \{p, q, r\} \), \( \tau_1 = \{0 \_X, 1 \_X, \lambda\} \) where \( \lambda(a) = 0.4 \), \( \lambda(b) = 0.7 \), \( \lambda(c) = 0.3 \) and \( \tau_2 = \{0 \_Y, 1 \_Y, \mu\} \) where \( \mu(p) = 0.7 \), \( \mu(q) = 0.7 \), \( \mu(r) = 0.8 \). Here \( f : X \to Y \) defined by \( f(a) = p \), \( f(b) = q \), \( f(c) = r \). Then \( f \) is regular generalized fuzzy homeomorphism but \( f \) is not generalized fuzzy homeomorphism.

**Theorem 6.10.** Every generalized regular fuzzy homeomorphism is generalized fuzzy homeomorphism.

**Proof.** Straight forward.

**Remark 6.11.** The converse of the above theorem may not be true as shown in the following example.

**Example 6.12.** Let \( X = \{a, b, c\} \), \( Y = \{p, q, r\} \), \( \tau_1 = \{0 \_X, 1 \_X, \lambda\} \) where \( \lambda(a) = 0.4 \), \( \lambda(b) = 0.6 \), \( \lambda(c) = 0.8 \), \( \tau_2 = \{0 \_Y, 1 \_Y, \mu\} \) where \( \mu(p) = 0.5 \), \( \mu(q) = 0.7 \), \( \mu(r) = 0.3 \).
Here $f : X \to Y$ defined by $f(a) = p$, $f(b) = q$, $f(c) = r$. Thus $f$ is generalized fuzzy homeomorphism but $f$ is not generalized regular fuzzy homeomorphism.

**Definition 6.13.** A bijection $f : X \to Y$ is called generalized fuzzy closed homeomorphism if $f$ is fuzzy generalized closed irresolute and its inverse $f^{-1}$ is also fuzzy generalized closed irresolute.

**Theorem 6.14.** Every generalized fuzzy closed homeomorphism is generalized fuzzy homeomorphism.

**Remark 6.15.** However the converse of the above theorem is not true as shown in the following example.

**Example 6.16.** Let $X = \{a, b, c\}, Y = \{p, q, r\}, \tau_1 = \{0_x, 1_x, \lambda\}$ where $\lambda(a) = 0.4$, $\lambda(b) = 0.6$, $\lambda(c) = 0.8$. $\tau_2 = \{0_y, 1_y, \mu\}$ where $\mu(p) = 0.5$, $\mu(q) = 0.7$, $\mu(r) = 0.3$. Here $f : X \to Y$ defined by $f(a) = p$, $f(b) = q$, $f(c) = r$. Then $f$ is generalized fuzzy homeomorphism. But $\{(p, 0.4), (q, 0.5), (r, 0.4)\}$ is a gfc in $(Y, \tau_2)$ but it is not gfc in $(X, \tau_1)$. So $f$ is not generalized fuzzy closed homeomorphism.

**Definition 6.17.** A bijection $f : X \to Y$ is called regular generalized fuzzy closed homeomorphism if $f$ is regular generalized fuzzy closed irresolute and its inverse $f^{-1}$ is also regular generalized fuzzy closed irresolute.

**Theorem 6.18.** Every regular generalized fuzzy closed homeomorphism is regular generalized fuzzy homeomorphism.

**Remark 6.19.** However the converse of the above theorem may not be true as shown in the following example.

**Example 6.20.** Let $X = \{a, b, c\}, Y = \{p, q, r\}, \tau_1 = \{0_x, 1_x, \lambda\}$ where $\lambda(a) = 0.4$, $\lambda(b) = 0.6$, $\lambda(c) = 0.3$ and $\tau_2 = \{0_y, 1_y, \mu\}$ where $\mu(p) = 0.7$, $\mu(q) = 0.7$, $\mu(r) = 0.8$. Here $f : X \to Y$ defined by $f(a) = p$, $f(b) = q$, $f(c) = r$. Here $f$ is generalized fuzzy homeomorphism. But $\{(p, 0.2), (q, 0.7), (r, 0.1)\}$ is a rgfc in $(Y, \tau_2)$. Though it is not in $rgfc(X, \tau_1)$. So $f$ is not regular generalized fuzzy closed homeomorphism.

**Definition 6.21.** A bijection $f : X \to Y$ is called generalized regular fuzzy closed homeomorphism if $f$ is fuzzy generalized regular closed irresolute and its inverse $f^{-1}$ is also fuzzy generalized regular closed irresolute.

**Theorem 6.22.** Every generalized regular fuzzy closed homeomorphism is generalized regular fuzzy homeomorphism.
Remark 6.23. However the converse of the above theorem may not be true as shown in the following example.

Example 6.24. Let $X = \{a, b, c\}, Y = \{p, q, r\}, \tau_X = \{0_X, 1_X, \lambda\}$ where $\lambda(a) = 1, \lambda(b) = 0, \lambda(c) = 0$ and $\tau_Y = \{0_Y, 1_Y, \mu\}$ where $\mu(p) = 0, \mu(q) = 1, \mu(r) = 0$. Here $f : X \rightarrow Y$ defined by $f(a) = p, f(b) = q, f(c) = r$. Then $f$ is generalized fuzzy homeomorphism. Thus $\{(p,1), (q,0), (r,0)\}$ is a grfc in $(Y, \tau_Y)$ but it is not grfc in $(X, \tau_X)$. So $f$ is not generalized regular fuzzy closed homeomorphism.

References

(\varepsilon, \varepsilon \lor \neg \eta)\text{-fuzzy Subalgebras of Lattice Implication Algebras}

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Abstract:
In this paper we propose the concept of \((\varepsilon, \varepsilon \lor \neg \eta)\text{-fuzzy subalgebras of lattice implication algebras (LIA}s)\), and discuss its equivalent characterization. Then, some related properties are investigated, such as the properties about lattice implication homomorphism of \((\varepsilon, \varepsilon \lor \neg \eta)\text{-fuzzy subalgebras of LIA}s\) are given. Finally, new operations of \((\varepsilon, \varepsilon \lor \neg \eta)\text{-fuzzy subalgebras of LIA}s\) are introduced and also discuss its related properties.

Keywords:
Lattice implication algebra, \((\varepsilon, \varepsilon \lor \neg \eta)\text{-fuzzy subalgebra, homomorphism.}

1. Introduction

In the real world there exists a lot of uncertainty information, such as incomplete information, incomparable information, fuzzy information, etc. with the development of science and technology, there are large amounts of information, how the machine was used to simulate the human brain to effectively deal with the uncertainty information was attention widely by the people. Intelligent information processing is one important research direction in artificial intelligence. Information processing dealing with certain information is based on the classical logic. However, non-classical logics including logics behind fuzzy reasoning handle information with various facets of uncertainty such as fuzziness, randomness, etc. Therefore, non-classical logic have become as a formal and useful tool for computer science to deal with uncertain information. Many-valued logic, a great extension and development of classical logic, has always been an important direction in non-classical logic. In 1993, in order to investigate a many-valued logical system whose prepositional values is given in a lattice, Xu established the lattice implication algebra by combining lattice and implication algebra, and discussed many useful structures \([1-3]\). Its related properties and structures are further studied \([4, 17, 18]\).
The concept of fuzzy set was introduced by Zadeh [8, 9]. Since then this idea has been applied to other algebraic structures [6, 13-16]. Rosenfeld inspired the fuzzification of algebraic structure and introduced the notion of fuzzy subgroup [11]. Pu and Liu given the idea of fuzzy point and ‘belongingness’ and ‘quasi-coincidence’ with a fuzzy set [12]. Jun et al [19] introduced the concept of $(e, e \lor q)$-fuzzy implicative filter of a lattice implication algebra. Zhan et al [7] investigated this kind of fuzzy implicative filters. Liu Yi et al [20, 21] further studied this kind of fuzzy implicative filters. Recently, Ruijuan Lv [5] introduced the concept of $(e, e \lor q)$-fuzzy subalgebras of lattice implication algebras and discussed some of their related properties.

2. Preliminaries

In this section, we give some notions, definitions and basic results which will be needed in the following discussion.

**Definition 2.1** [3] (Lattice implication algebra). Let $(L, \lor, \land, O, I)$ be a bounded lattice with an order-reversing involution “‘”, $I$ and $O$ the greatest and the smallest element of $L$ respectively, and $\rightarrow: L \times L \rightarrow L$ be a mapping $(L, \lor, \land, ‘, \rightarrow, O, I)$ is called a lattice implication algebra if the following conditions hold for any $x, y, z \in L$:

$$(I_1) \quad x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z);$$

$$(I_2) \quad x \rightarrow x = I;$$

$$(I_3) \quad x \rightarrow y = y' \rightarrow x';$$

$$(I_4) \quad x \rightarrow y = y \rightarrow x = I \text{ implies } x = y;$$

$$(I_5) \quad (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x;$$

$$(I_6) \quad (x \lor y) \rightarrow z = (x \rightarrow z) \land (y \rightarrow z);$$

$$(I_7) \quad (x \land y) \rightarrow z = (x \rightarrow z) \lor (y \rightarrow z).$$

**Theorem 2.1** [3]. Let $L$ be a lattice implication algebra, then for any $x, y, z \in L$:

1. If $I \rightarrow x = I$, then $x = I$;
2. $I \rightarrow x = x$ and $x \rightarrow O = x'$;
3. $O \rightarrow x = I$ and $x \rightarrow I = I$;
4. $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = I$;
5. $(x \rightarrow y) \rightarrow x' = (y \rightarrow x) \rightarrow y'$;
6. $x \land y = ((x \rightarrow y) \rightarrow x')'$;
7. $x \lor y = ((x \rightarrow y) \rightarrow y)$.
Definition 2.2 [3] (Lattice implication subalgebra). Let $L$ be a lattice implication algebra. $S \subseteq L$ is called a lattice implication subalgebra of $L$, if the following conditions hold:

1. $(S, \lor, \land, ^{'} )$ is bounded sublattice of $(L, \lor, \land)$ with an order-reversing involution $^{'}$;
2. If $x, y \in S$, then $x \rightarrow y \in S$.

For a fuzzy subset $F$ of $L$ and $t \in (0, 1]$, the crisp set $U(F; t) = \{ x \in L | F(x) \geq t \}$ is called the level subset of $F$.

A fuzzy subset $F$ of $L$ the form

$$F(y) = \begin{cases} t(\neq 0) & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support $x$ and value $t$ and is denoted by $x_{t}$.

A fuzzy point $x_{t}$ is said to belong to (resp. be quasi-coincident with) a fuzzy subset $F$, written by $x_{t} \in F$ (resp., $x_{t} \sim qF$) if $F(x) \geq t$ (resp., $F(x) + t > 1$). If $x_{t} \in F$ or $x_{t} \sim qF$, then we write $x_{t} \in \nu qF$. If $F(x) < t$ (resp., $F(x) + t \leq 1$), then we call $x_{t} \in F$ (resp., $x_{t} \sim qF$).

Definition 2.3 [2]. Let $L$ be a lattice implication algebra, $A \subseteq L$, then $A$ is a subalgebra of $L$ if it satisfies the following conditions:

1. $O \in A$;
2. For any $x, y \in A$, implies $x \rightarrow y \in A$.

Definition 2.4 [2]. Let $L$ be a lattice implication algebra, $A \subseteq F(L)$, then $A$ is a fuzzy subalgebra of $L$ if it satisfies the following conditions:

1. $A(I) = A(O)$;
2. For any $x, y \in A$, $A(x \rightarrow y) \geq \min \{ A(x), A(y) \}$.

Theorem 2.2 [2]. Let $L$ be a lattice implication algebra, $A$ is a fuzzy subalgebra of $L$, then for any $x, y \in L$,

1. $A(I) = A(O) \geq A(x)$;
2. $A(x) = A(x^{'})$;
3. $A(x \lor y) \geq \min \{ A(x), A(y) \}$;
4. $A(x \land y) \geq \min \{ A(x), A(y) \}$.

Theorem 2.3 [4]. Let $L_{1}, L_{2}$ be lattice implication subalgebras of $L$, then $L_{1} \cap L_{2}$ is also lattice implication subalgebra of $L$. 
Definition 2.5 [5]. Let $L$ be a lattice implication algebra, $A \subseteq F(L)$, then $A$ is an $(\varepsilon, \varepsilon \lor q)$-fuzzy subalgebra of $L$ if it satisfies the following conditions hold for any $x, y \in L$, $r, t \in (0, 1)$:

1. $x_r \in A \Rightarrow x'_r \in \lor q A$;
2. $x, y_t \in A \Rightarrow (x \rightarrow y)_{\min[r, t]} \in \lor q A$.

Theorem 2.4 [5]. Let $L$ be a lattice implication algebra, $A \subseteq F(L)$, then $A$ is an $(\varepsilon, \varepsilon \lor q)$-fuzzy subalgebra of $L$ if and only if the following conditions hold for any $x, y \in L$, $r, t \in (0, 1)$:

1. $A(x') \geq \min \{A(x), 0.5\}$;
2. $A(x \rightarrow y) \geq \min \{A(x), A(y), 0.5\}$.

Definition 2.6 [6]. Let $A, B$ are fuzzy sets of $X$ and $Y$, definition the mapping $A \times B : X \times Y \rightarrow [0, 1]$, $(A \times B)(x, y) = \min \{A(x), A(y)\}$, $\forall (x, y) \in X \times Y$, then $A \times B$ is the fuzzy set of $X \times Y$, called direct product of $A \times B$.

3. The $(\varepsilon, \varepsilon \lor q)$-fuzzy subalgebras of lattice implication algebras

In this section, we introduce the concepts of $(\varepsilon, \varepsilon \lor q)$-fuzzy subalgebras of lattice implication algebras, and discuss its equivalent characterization as well as some properties.

Definition 3.1. Let $L$ be a lattice implication algebra, $A \subseteq F(L)$, then $A$ is an $(\varepsilon, \varepsilon \lor q)$-fuzzy subalgebra of $L$, if it satisfies the following conditions hold for any $x, y \in L$, $r, t \in (0, 1)$:

1. $x'_r \in A \Rightarrow x_r \in \lor q A$;
2. $(x \rightarrow y)_{\min[r, t]} \in A \Rightarrow x, y \in \lor q A$ or $y_t \in \lor q A$.

Theorem 3.1. Let $L$ be a lattice implication algebra, $A \subseteq F(L)$, then $A$ is an $(\varepsilon, \varepsilon \lor q)$-fuzzy subalgebra of $L$ if and only if the following conditions hold for any $x, y \in L$, $r, t \in (0, 1)$:

3. $\max \{A(x'), 0.5\} \geq A(x)$;
4. $\max \{A(x \rightarrow y), 0.5\} \geq \min \{A(x), A(y)\}$. 
Proof. (1) $\Rightarrow$ (3) If there exists $x \in L$ such that $\max \{ A(x'), 0.5 \} < A(x) = r$, then $0.5 < r \leq 1$, $x' \in A$ but $x \notin A$. By (1), we have $x, \overline{q}A$, and whence, $A(x) \geq r$, $A(x) + r \leq 1$. Thus $r < 0.5$, a contradiction.

(3) $\Rightarrow$ (1) Let $x' \notin A$, then $A(x') < r$.

(a) If $A(x') \geq A(x)$, then $A(x) < r$, and so $x \notin A$. This proves that $x, \overline{q}A$.

(b) If $A(x') < A(x)$, then by (3), we have $0.5 \geq A(x)$. Putting $x, \in A$, then $0.5 \geq A(x) \geq r$, it follows that $A(x) + r \leq 1$ and so $x, \overline{q}A$. This proves that $x, \overline{q}A$.

(2) $\Rightarrow$ (4). If there exists $x, y \in L$ such that $\max \{ A(x \rightarrow y), 0.5 \} < t = \min \{ A(x), A(y) \}$, then $0.5 < t \leq 1$, $(x \rightarrow y) \in A$, but $x, \notin A$, $y, \in A$. By (2), we have $x, \overline{q}A$ or $y, \overline{q}A$. Then $(A(x) \geq t$ and $A(x) + t \leq 1$) or $(A(y) \geq t$ and $A(y) + t \leq 1$). This implies that $t \leq 0.5$, a contradiction.

(4) $\Rightarrow$ (2). (Let $(x \rightarrow y)_{\min \{r,t\}} \in A$. Then $A(x \rightarrow y) < \min \{r,t\}$.

(a) If $A(x \rightarrow y) \geq \min \{A(x), A(y)\}$, then $\min \{A(x), A(y)\} < \min \{r,t\}$ and consequently, $A(x) < r$ or $A(y) < t$. It follows that $x, \in A$ or $y, \in A$. Thus, $x, \in \overline{q}A$ or $y, \in \overline{q}A$.

(b) If $A(x \rightarrow y) < \min \{A(x), A(y)\}$, then by (4), we have $0.5 \geq \min \{A(x), A(y)\}$. Putting $x, \in A$ and $y, \in A$. Then $t \leq A(y) \leq 0.5$ or $r \leq A(x) \leq 0.5$, it follows that $t + A(y) \leq 1$ or $r + A(x) \leq 1$, and thus $x, \in \overline{q}A$ or $y, \in \overline{q}A$.

Theorem 3.2. Let $L$ be a lattice implication algebra, $A \subseteq F(L)$, then $A$ is an $(\overline{\oplus}, \ominus)$-fuzzy subalgebra of $L$ if $A$ is a fuzzy subalgebra of $L$.

Proof. This theorem is an immediate consequence of Definition 2.2 and Theorem 3.1. In general, the $(\overline{\oplus}, \ominus)$-fuzzy subalgebra of $L$ is not a fuzzy subalgebra of $L$, for Example 3.1 (1).

Theorem 3.3. Let $L$ be a lattice implication algebra, $A \subseteq F(L)$, then $A$ is a fuzzy subalgebra of $L$ if $A$ is an $(\overline{\oplus}, \ominus)$-fuzzy subalgebra of $L$ and $0.5 < A(O) \leq 1$.

Proof. Let $A$ is an $(\overline{\oplus}, \ominus)$-fuzzy subalgebra of $L$, then for any $x, y \in L$, such that $\max \{A(x'), 0.5\} \geq A(x)$. Hence $\max \{A(O), 0.5\} \geq A(I)$, since $0.5 < A(O) \leq 1$, then $A(O) \geq A(I)$. And $\max \{A(I), 0.5\} \geq A(O)$, then $A(O) \leq A(I)$. Hence $A(O) = A(I)$. And for any $x, y \in L$, such that $\max \{A(x \rightarrow y), 0.5\} \geq \min \{A(x), A(y)\}$. If $A(x \rightarrow y) < \min \{A(x), A(y)\}$, then $0.5 \geq \min \{A(x), A(y)\}$. By $0.5 < A(O) = A(I) \leq 1$, a con-
tradiction. Then $A(x \rightarrow y) \geq \min \{A(x), A(y)\}$. This proves that $A$ is a fuzzy subalgebra of $L$.

**Example 3.1.** Let $L = \{O, a, b, I\}$, the Hasse diagram of $L$ be defined as Fig. 1 and operators of $L$ be defined in Table 1, then $(L, \lor, \land, \cdot, \rightarrow)$ is a lattice implication algebra.

![Hasse diagram](image)

**Fig. 1.** Hasse diagram of $L = \{O, a, b, I\}$

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(2) Define a fuzzy subset \( C \) of \( L \) by \( C(O) = 0.6 \), \( C(a) = 0.8 \), \( C(b) = 0.3 \) \( C(I) = 0.7 \), because for \( \max \{C(b), 0.5\} = 0.5 < 0.8 = C(a) \), then it can be easily verified that \( B \) is not an \( (\mathcal{E}, \mathcal{E} \vee \mathcal{T}) \)-fuzzy subalgebra of \( L \). Because for \( C(I) = 0.7 \neq 0.6 = C(O) \), then \( C \) is not a fuzzy subalgebra of \( L \) by Definition 2.4.

(3) Define a fuzzy subset \( B \) of \( L \) by \( B(O) = 0.6 \), \( B(a) = 0.3 \), \( B(b) = 0.3 \), \( B(I) = 0.6 \), then it can be easily verified that \( B \) is an \( \mathcal{E} \)-fuzzy subalgebra of \( L \). \( B \) is a fuzzy subalgebra of \( L \) by Definition 2.4.

**Theorem 3.4.** Let \( L \) be a lattice implication algebra, \( A \subseteq F(L) \), \( A \) is an \( \mathcal{E} \)-fuzzy subalgebra of \( L \), for any \( x, y \in L \), then the following statements hold.

(1) \( \max \{A(O), 0.5\} \geq A(x) \);

(2) \( \max \{A(x \vee y), 0.5\} \geq \min \{A(x), A(y)\} \);

(3) \( \max \{A(x \wedge y), 0.5\} \geq \min \{A(x), A(y)\} \).

*Proof.* (1) \( A \) is an \( \mathcal{E} \)-fuzzy subalgebra of \( L \), for any \( x, y \in L \), then \( \max \{A(O), 0.5\} \geq A(I) \), hence \( \max \{A(O), 0.5\} \geq \max \{A(I), 0.5\} \geq \max \{A(x \rightarrow x), 0.5\} \geq \min \{A(x), A(x)\} \geq A(x) \).

(2) By Theorem 3.1, \( \max \{A(x \vee y), 0.5\} = \max \{A((x \rightarrow y) \rightarrow y), 0.5\} = \max \{A(x \rightarrow y), 0.5\} \geq \max \{\min \{A(x \rightarrow y), A(y)\}, 0.5\} = \min \{\max \{A(x \rightarrow y), 0.5\}, \max \{A(y), 0.5\}\} \geq \min \{\min \{A(x), A(y)\}, \max \{A(y), 0.5\}\} = \min \{A(x), A(y)\} \).

(3) By Theorem 3.1 and Theorem 3.4 (2), \( \max \{A(x \wedge y), 0.5\} \geq A(x \wedge y)' \), then \( \max \{A(x \wedge y), 0.5\} \geq \max \{A(x)' \vee y', 0.5\} = \max \{A(x \rightarrow y)', 0.5\} \geq \min \{A(x'), A(y')\} \), then \( \max \{A(x \rightarrow y)', 0.5\} \geq \min \{\max \{A(x'), 0.5\}, \max \{A(y'), 0.5\}\} \geq \min \{A(x), A(y)\} \).

**Theorem 3.5.** Let \( L \) be a lattice implication algebra, \( A \subseteq F(L) \), then \( A \) is an \( \mathcal{E} \)-fuzzy subalgebra of \( L \) if and only if for any \( t \in (0.5, 1] \), \( A_t = \{x \in L | A(x) \geq t\} \) is a subalgebra of \( L \).

*Proof.* Let \( A \) be an \( \mathcal{E} \)-fuzzy subalgebra of \( L \), for any \( x, y \in A_t \), \( t \in (0.5, 1] \). By Theorem 3.4, there exist \( x_0 \in A_t \), then \( \max \{A(O), 0.5\} \geq A(x_0) = t \). It follows that \( A(O) \geq A(x_0) \), and hence \( O \in A_t \). Putting \( \max \{A(x \rightarrow y), 0.5\} \geq \min \{A(x), A(y)\} = t \),
then \( A(x \rightarrow y) \geq t \), and hence \( x \rightarrow y \in A_i \). It follows that \( A_i \) is a subalgebra of \( L \) by Definition 2.3.

Conversely, if \( A_i \) is a subalgebra of \( L \), for any \( t \in (0.5,1] \). Then \( A \) is an \((\Xi,\Xi \vee \bar{\eta})\)-fuzzy subalgebra of \( L \) is an immediate consequence of Definition 2.1 and Theorem 3.1.

**Theorem 3.6.** Let \( L \) be a lattice implication algebra, \( A,B \subseteq F(L) \), \( A, B \) are \((\Xi,\Xi \vee \bar{\eta})\)-fuzzy subalgebras of \( L \), then \( A \cap B \) is an \((\Xi,\Xi \vee \bar{\eta})\)-fuzzy subalgebra of \( L \).

Proof. \( A, B \) are \((\Xi,\Xi \vee \bar{\eta})\)-fuzzy subalgebras of \( L \). By Theorem 3.1, for any \( x,y \in L \), 
\[
\max\{ (A \cap B)(x'), 0.5 \} = \max\{ A(x'), 0.5 \} \wedge \max\{ A(x'), 0.5 \} \geq A(x) \cap B(x) = (A \cap B)(x),
\]
\[
\max\{ (A \cap B)(x \rightarrow y), 0.5 \} = \max\{ A(x \rightarrow y), 0.5 \} \wedge \max\{ B(x \rightarrow y), 0.5 \} \geq \min\{ A(x), A(y) \} \wedge \min\{ B(x), B(y) \} = \min\{ (A \cap B)(x), (A \cap B)(y) \},
\]
then \( A \cap B \) is an \((\Xi,\Xi \vee \bar{\eta})\)-fuzzy subalgebra of \( L \).

**Corollary 3.1.** Let \( L \) be a lattice implication algebra, \( A^i \subseteq F(L) \) \( i = 1,2,\cdots,n \). \( A^i \) is \((\Xi,\Xi \vee \bar{\eta})\)-fuzzy subalgebras of \( L \) \( i = 1,2,\cdots,n \), then \( \cap A^i \) is an \((\Xi,\Xi \vee \bar{\eta})\)-fuzzy subalgebra of \( L \) \( i = 1,2,\cdots,n \).

**Theorem 3.7.** Let \( L_1, L_2 \) be lattice implication algebras, \( f:L_1 \rightarrow L_2 \) is a lattice implication isomorphism of \( L_1 \) to \( L_2 \), \( A \in L_1 \). If \( A \) is an \((\Xi,\Xi \vee \bar{\eta})\)-fuzzy subalgebras of \( L_1 \), then \( f(A) \) is an \((\Xi,\Xi \vee \bar{\eta})\)-fuzzy subalgebras of \( L_2 \).

Proof. Since \( f:L_1 \rightarrow L_2 \) is a lattice implication isomorphism of \( L_1 \) to \( L_2 \), then for any \( y_1,y_2 \in L_2 \), there exists \( x_1,x_2 \in L_1 \), such that \( f(x_1) = y_1 \), \( f(x_2) = y_2 \). We have 
\[
\max\{ f(A)(y_1'), 0.5 \} = \max\{ \vee f(x_1) \wedge A(x_1'), 0.5 \} \geq \vee f(x_1) \wedge A(x_1) = f(A)(y_1).
\]
If there exists \( y_1,y_2 \in L_2 \), such that \( \max\{ f(A)(y_1 \rightarrow y_2), 0.5 \} \leq \min\{ f(A)(y_1), f(A)(y_2) \} \), thus \( f(A)(y_1 \rightarrow y_2) < f(A)(y_1) = \vee f(x_1) \wedge A(x_1) \), \( f(A)(y_1 \rightarrow y_2) < f(A)(y_2) = \vee f(x_2) \wedge A(x_2) \), then \( \max\{ A(x_1 \rightarrow x_2), 0.5 \} \leq \vee f(x_1) \wedge f(x_2) \wedge A(x) = f(A)(y_1 \rightarrow y_2) \). This is a contradiction by Theorem 3.1, thus \( \max\{ f(A)(y_1 \rightarrow y_2), 0.5 \} \geq \min\{ f(A)(y_1), f(A)(y_2) \} \).

Therefore, \( f(A) \) is the \((\Xi,\Xi \vee \bar{\eta})\)-fuzzy subalgebras of \( L_2 \).

**Theorem 3.8.** Let \( L_1, L_2 \) are lattice implication algebras, \( f:L_1 \rightarrow L_2 \) is a lattice implication homomorphism of \( L_1 \) to \( L_2 \), \( A \in L_1 \), \( B \in L_2 \). If \( B \) is an \((\Xi,\Xi \vee \bar{\eta})\)-fuzzy subalgebras of \( L_2 \), then \( f^{-1}(B) \) is an \((\Xi,\Xi \vee \bar{\eta})\)-fuzzy subalgebras of \( L_1 \).
Proof. For any $x_i, x'_i, x_2 \in L_i$, such that $f(x_i) = y_i, f(x'_i) = y'_i$, because of $B$ is the $(\varepsilon, \exists \vee q \exists)$-fuzzy subalgebras of $L_2$, thus by Theorem 3.1, we have $\max \{ f^{-1}(B)(x'_i), 0.5 \} = \max \{ f^{-1}(B)(y'_i), 0.5 \} \leq f^{-1}(B)(y_i) = B(f(x_i)) = f^{-1}(B)(x_i)$. max $\{ f^{-1}(B)(x_i \rightarrow x_2), 0.5 \} = \max \{ f^{-1}(B)(x_i \rightarrow x_2), 0.5 \} \leq \min \{ B(f(x_i)), B(f(x'_i)) \} = \min \{ f^{-1}(B)(x_i), f^{-1}(B)(x'_i) \}$. Therefore, $f^{-1}(B)$ is an $(\varepsilon, \exists \vee q \exists)$-fuzzy subalgebras of $L_i$.

4. The direct product of $(\varepsilon, \exists \vee q \exists)$-fuzzy subalgebras of lattice implication algebras

In this section, we introduce the concepts of the direct product of $(\varepsilon, \exists \vee q \exists)$-fuzzy subalgebras of lattice implication algebras, and discuss their related properties.

Definition 4.1. Let $L_1, L_2$ are lattice implication algebras, $A, B$ are $(\varepsilon, \exists \vee q \exists)$-fuzzy subalgebras of $L_1$ and $L_2$, definition the mapping $A \times B : X \times Y \rightarrow [0,1]$,

$$(A \times B)(x, y) = \min \{ A(x), B(y) \}, \forall (x, y) \in L_1 \times L_2,$$

$A \times B$ is called the direct product of $(\varepsilon, \exists \vee q \exists)$-fuzzy subalgebra of $L_1$ and $L_2$.

Theorem 4.1. Let $L_1, L_2$ are lattice implication algebras, $A, B$ are $(\varepsilon, \exists \vee q \exists)$-fuzzy subalgebras of $L_1$ and $L_2$, then $A \times B$ is an $(\varepsilon, \exists \vee q \exists)$-fuzzy subalgebra of $L_1 \times L_2$.

Proof. $A, B$ are $(\varepsilon, \exists \vee q \exists)$-fuzzy subalgebras of $L_1$ and $L_2$, for any $(x, y) \in L_1 \times L_2$, then by Theorem 3.1 and Definition 3.2, we have $\max \{ (A \times B)(x, y, y'), 0.5 \} = \max \{ \min \{ A(x'), B(y) \}, 0.5 \} \leq \min \{ A(x'), B(y) \}$.

$\max \{ (A \times B)(x \rightarrow y), 0.5 \} \geq \min \{ A(x \rightarrow y), 0.5 \} \leq \min \{ A(x \rightarrow y), 0.5 \} \geq \min \{ A(x), A(y), B(x), B(y) \} \leq \min \{ A(x), A(y), B(x), B(y) \} = \min \{ A(x), A(y), B(x), B(y) \}$. Then $A \times B$ is an $(\varepsilon, \exists \vee q \exists)$-fuzzy subalgebra of $L_1 \times L_2$ by Theorem 3.1.

Corollary 4.1. Let $L_1, L_2, \cdots, L_n$ are lattice implication algebras, $A^1, A^2, \cdots, A^n$ are $(\varepsilon, \exists \vee q \exists)$-fuzzy subalgebras of $L_1, L_2, \cdots, L_n$, then $A^1 \times A^2 \times \cdots \times A^n$ is an $(\varepsilon, \exists \vee q \exists)$-fuzzy subalgebra of $L_1 \times L_2 \times \cdots \times L_n$.
5. Conclusion

In this paper, we firstly introduced the concepts of $(\Xi, \Xi \lor \varphi)$-fuzzy subalgebras of lattice implication algebras, and further discussed its equivalent characterization. Then we studied homomorphic properties of $(\Xi, \Xi \lor \varphi)$-fuzzy subalgebras of lattice implication algebras. Finally we researched the intersection operations and direct product operations of $(\Xi, \Xi \lor \varphi)$-fuzzy subalgebras of lattice implication algebras.

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